Shape Prior Segmentation Utilizing Cliques of Points and Geodesic Active Contours *

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Abstract

We propose a variational invariant shape prior signature utilizing cliques of points coupled with geodesic active contours (snakes) for shape prior segmentation and image disocclusion problems. The model will be shown to be advantageous over the standard geodesic active contours when segmenting images that contain disoclussions of many scales as well as significant clutter.

1 Proposed Signature: Pairwise Differences of Cliques of Points

We consider the following energy:

$$\inf_{\phi} \left\{ E(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N) = \sum_{i,j} \left(\|\mathbf{p}_i - \mathbf{p}_j\|^2 - \|\mathbf{r}_i - \mathbf{r}_j\|^2 \right)^2 \right\} \tag{1}$$

where $\{\mathbf{p}_i\}$ denote chosen grid points lying on the contour defining the shape of the evolving contour defined by ϕ and $\{\mathbf{r}_i\}$ are those lying on the reference contour ψ.

The main idea here is to combine this invariant signature with Geodesic Active Contours (GAC) (cite) utilizing an edge detector to stop the evolving contour. One such example is:

$$g(|\nabla f(x,y)|) = \frac{1}{1 + |\nabla G_\sigma(x,y) * f(x,y)|^p}, \quad p \geq 1. \tag{2}$$

Here, $G_\sigma(x,y) * f(x,y)$ is a smoothed out version of the given image f, $*$ denotes the convolution operator, and G_σ is a Gaussian.

Let us now compute variations of E with respect to the evolving points $\mathbf{p}_l$. Here, we assume that only the $\mathbf{p}_l$ for fixed l varies in time. Thus we compute the time derivative of E:

$$\begin{aligned} \frac{d}{dt}E &= \frac{d}{dt} \sum_{i,j} \left(\|\mathbf{p}_i - \mathbf{p}_j\|^2 - \|\mathbf{r}_i - \mathbf{r}_j\|^2 \right)^2 && (3) \\ &= 4 \sum_{i,j} \left(\|\mathbf{p}_i - \mathbf{p}_j\|^2 - \|\mathbf{r}_i - \mathbf{r}_j\|^2 \right) \frac{d}{dt} \|\mathbf{p}_i - \mathbf{p}_j\|^2. && (4) \end{aligned}$$

*

Now,

$$\frac{d}{dt}\|\mathbf{p}_i - \mathbf{p}_j\|^2 \;=\; \frac{d}{dt}\left(\mathbf{d}_{i,j}^2\right) \;=\; 2(\delta_{i,l} - \delta_{j,l}) < \dot{\mathbf{p}}_l,\, \mathbf{p}_i - \mathbf{p}_j > . \tag{5}$$

Thus, setting:

$$B(i,j,l) \;=\; (\delta_{i,l} - \delta_{j,l})(\mathbf{p}_i - \mathbf{p}_j) \tag{6}$$

and

$$A(i,j) = \left(\|\mathbf{p}_i - \mathbf{p}_j\|^2 - \|\mathbf{r}_i - \mathbf{r}_j\|^2\right) \tag{7}$$

we arrive at:

$$\frac{d}{dt}E \;=\; \sum_{i,j} A(i,j)\,\langle B(i,j,l),\, \dot{\mathbf{p}}_l\rangle\,. \tag{8}$$

Hence, gradient descent is given by the following ODE system:

$$\dot{\mathbf{p}}_l \;=\; \sum_{i,j} A(i,j)B(i,j,l). \tag{9}$$

Setting $C(l) = \sum_{i,j} A(i,j)B(i,j,l)$, we see that this term $C(l)$ can be computed fast since most of the terms are 0; the only choice of indices that contribute are:

$$\{(i,j) : i = l \,\text{and}\, j \neq l\} \cup \{(i,j) : j = l \,\text{and}\, i \neq l\}\,. \tag{10}$$

Thus, $C(l)$ reduces to:

$$C(l) = \sum_{i \neq l} A(i,j)(\mathbf{p}_j - \mathbf{p}_l) \tag{11}$$

and the gradient descent on the proposed energy reduces to:

$$\dot{\mathbf{p}}_l \;=\; \sum_{i \neq l} A(i,j)(\mathbf{p}_j - \mathbf{p}_l). \tag{12}$$

2 Snakes and Geodesic Active Contours Utilizing the Proposed Signature

Let the curve $\mathcal{C}(p) = (x(p), y(p))$, where $p \in [0,1]$ is an arbitrary parametrization. The Snake model is defined as the energy functional:

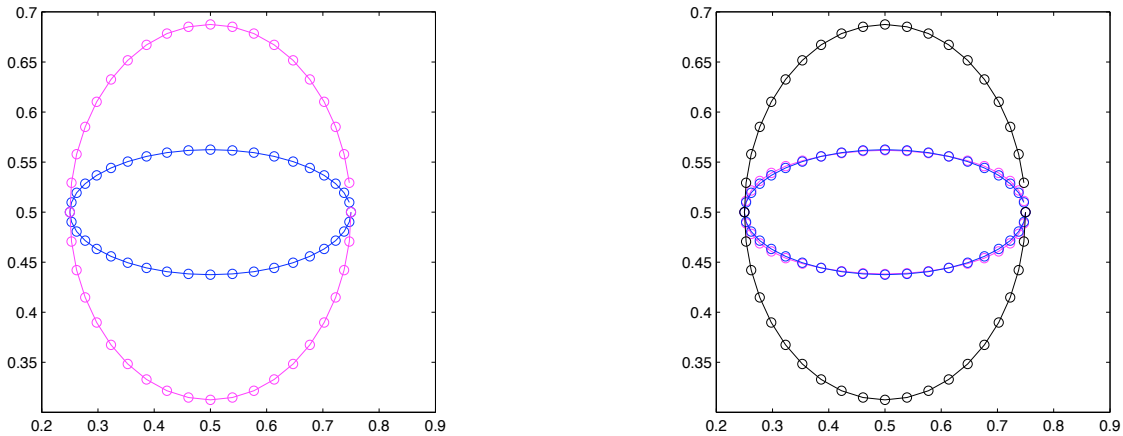
$$S(\mathcal{C}) = \int_0^1 \left(|\mathcal{C}_p|^2 + \alpha|\mathcal{C}_{pp}|^2 + \beta g(\mathcal{C})\right) dxdy. \tag{13}$$

Here, $\mathcal{C}(p) = (\partial_p x(p), \partial_p y(p))$ and

$$g(x,y) = \frac{1}{1 + |\nabla(G_\sigma(x,y) * f(x,y))|^p}, \quad p \geq 1 \tag{14}$$

or just simply

$$g(x,y) = \frac{1}{1 + |\nabla f(x,y)|^p}, \quad p \geq 1 \tag{15}$$



reference (blue), initial (purple)

reference (blue), evolved (purple)
initial (black)

Figure 1: *Proposed Signature:*

where $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. Equation 13 has the following gradient descent:

$$\frac{d\mathcal{C}}{dt} = \mathcal{C}_{pp} - \alpha \mathcal{C}_{pppp} - \beta \nabla g. \tag{16}$$

Let us consider the reduced model:

$$S(\mathcal{C}) = \int_0^1 g(x(p), y(p)) |\mathcal{C}'(p)| dp. \tag{17}$$

The gradient descent for the above model (17) is given by:

$$\frac{d}{dt} \int_0^1 g(x(p), y(p)) |\mathcal{C}'(p)| dp. \tag{18}$$

where each of the points $\mathbf{x}(p) = \mathbf{x}(p, t) = (x(p, t), y(p, t))$ now depends on the parametrization variable p and a time variable t.

$$\frac{d}{dt} \int_0^1 g(x(p), y(p)) |\mathcal{C}'(p)| dp. \tag{19}$$

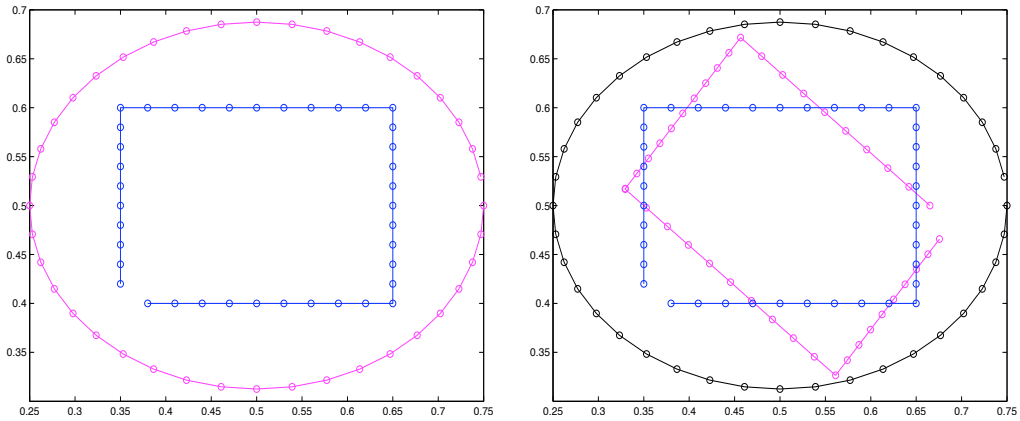
Here, the quantity $g(x(p), y(p))|\mathcal{C}'(p)| = g(x, y)\sqrt{x_p^2 + y_p^2}$. Thus

$$\frac{d}{dt}\left(g(x, y)\sqrt{x_p^2 + y_p^2}\right) = \left(\frac{d}{dt} g(x, y)\right)\sqrt{x_p^2 + y_p^2} + g(x, y)\left(\frac{d}{dt}\sqrt{x_p^2 + y_p^2}\right) \tag{20}$$

$$= \text{(I)} + \text{(II)}\,. \tag{21}$$

Now

$$\frac{d}{dt} g(x, y) = g_x\, x_t + g_y\, y_t = \vec{x}_t \cdot \nabla g \tag{22}$$



reference (blue), initial (purple)

reference (blue), evolved (purple)
initial (black)

Figure 2: *Proposed Signature:*

and

$$\frac{d}{dt}\sqrt{x_p^2+y_p^2} = \frac{1}{\sqrt{x_p^2+y_p^2}}\left(x_p(x_t)_p + y_p(y_t)_p\right) \tag{23}$$

$$= \frac{\vec{x}_p}{\sqrt{x_p^2+y_p^2}} \cdot (\vec{x}_t)_p \,. \tag{24}$$

Therefore,

$$\int_0^1 \frac{d}{dt}\left(g(x,y)\sqrt{x_p^2+y_p^2}\right)\, dp = \int_0^1 \left\{ \vec{x}_t \cdot \left((\nabla g)\sqrt{x_p^2+y_p^2}\right) + \frac{g\,\vec{x}_p}{\sqrt{x_p^2+y_p^2}} \cdot (\vec{x}_t)_p \right\} dp \,. \tag{25}$$

Integrating the second term in the above equation yields:

$$\int_0^1 \frac{d}{dt}\left(g(x,y)\sqrt{x_p^2+y_p^2}\right)\, dp = \int_0^1 \left((\nabla g)\sqrt{x_p^2+y_p^2} - \left(\frac{g\,\vec{x}_p}{\sqrt{x_p^2+y_p^2}}\right)_p \right) \cdot \vec{x}_t \, dp \,. \tag{26}$$

Which has the associated flow:

$$\vec{x}_t = \left(\frac{g\,\vec{x}_p}{\sqrt{x_p^2+y_p^2}}\right)_p - (\nabla g)\sqrt{x_p^2+y_p^2} \tag{27}$$

that can be solved component wise

$$x_t = \left(\frac{g\,x_p}{\sqrt{x_p^2+y_p^2}}\right)_p - (g_x)\sqrt{x_p^2+y_p^2} \tag{28}$$

$$y_t = \left(\frac{g\, y_p}{\sqrt{x_p^2 + y_p^2}} \right)_p - (g_y)\sqrt{x_p^2 + y_p^2}\,. \tag{29}$$

2.1 Numerical Method

The gradient descent equations for the geodesic active contour having the flow given by:

$$\vec{x}_t = \left(\frac{g\, \vec{x}_p}{\sqrt{x_p^2 + y_p^2}} \right)_p - (\nabla g)\sqrt{x_p^2 + y_p^2} \tag{30}$$

that can be solved component wise

$$x_t = \left(\frac{g\, x_p}{\sqrt{x_p^2 + y_p^2}} \right)_p - (g_x)\sqrt{x_p^2 + y_p^2} \tag{31}$$

$$y_t = \left(\frac{g\, y_p}{\sqrt{x_p^2 + y_p^2}} \right)_p - (g_y)\sqrt{x_p^2 + y_p^2}\,. \tag{32}$$

can be solved in the discrete setting. Let a given curve berepresented by the following grid points: $\vec{x}_1, \vec{x}_2, ..., \vec{x}_n$. We denote the forward difference operator by $D^+ u_i = \frac{u_{i+1} - u_i}{\Delta u}$ and the backward operator as $D^- u_i = \frac{u_i - u_{i-1}}{\Delta u}$. The centered operator $D^c = \frac{D^+ u_i + D^- u_i}{2}$.

Therefore, the discretization for 31 can be given by:

$$x_t = D^- \left(\frac{g(x_i, y_i)\, D^+ x_i}{\sqrt{(D^+ x_i)^2 + (D^+ y_i)^2 + \epsilon}} \right) - (D^c_{x_i} g(x_i, y_i))\sqrt{(D^c x_i)^2 + (D^c y_i)^2} \tag{33}$$

and the one for 32 can be given by:

$$y_t = D^- \left(\frac{g(x_i, y_i)\, D^+ y_i}{\sqrt{(D^+ x_i)^2 + (D^+ y_i)^2 + \epsilon}} \right) - (D^c_{y_i} g(x_i, y_i))\sqrt{(D^c x_i)^2 + (D^c y_i)^2}. \tag{34}$$

Of course the time derivative can be discretized as $\vec{x}_t = \frac{\vec{x}^{n+1} - \vec{x}^n}{dt}$.

Of course we need to add an additional penalty term to keep the points of the curve evenly distributed. Let $\vec{x}_j = (x_j, y_j)$. Then. $d_{j+1,j} = \|\vec{x}_{j+1} - \vec{x}_j\|$. Let us consider the penalty term that enforces that the points $\{\vec{x}_j\}$ on the curve be evenly distributed:

$$D(\vec{x}_1, \ldots, \vec{x}_n) = \sum_j \left(\mathbf{d}^2_{j+1,j} - \mathbf{d}^2_{j,j-1} \right)^2. \tag{35}$$

Let us once again calculate variations of (35) with respect to time t assuming that only $\vec{x}_l = \vec{x}_l(s,t)$ varies in time:

$$\frac{d}{dt} D(\vec{x}_1, \ldots, \vec{x}_n) = \frac{d}{dt} \sum_j \left(\mathbf{d}^2_{j+1,j} - \mathbf{d}^2_{j,j-1} \right)^2 \tag{36}$$

$$= 2 \sum_j \left(\mathbf{d}^2_{j+1,j} - \mathbf{d}^2_{j,j-1} \right) \frac{d}{dt} \left(\mathbf{d}^2_{j+1,j} - \mathbf{d}^2_{j,j-1} \right) \tag{37}$$

where

$$\frac{1}{2}\frac{d}{dt}\left(\mathbf{d}_{j+1,j}^2 - \mathbf{d}_{j,j-1}^2\right) = (\delta_{j+1,l} - \delta_{j,l})(< \dot{\vec{x}}_l, \vec{x}_{j+1} - \vec{x}_j >) - (\delta_{j,l} - \delta_{j-1,l})(< \dot{\vec{x}}_l, \vec{x}_j - \vec{x}_{j-1} >). \quad (38)$$

If we set:

$$F(j-1, j, j+1) = \left(\mathbf{d}_{j+1,j}^2 - \mathbf{d}_{j,j-1}^2\right) \quad (39)$$

and

$$\mathbf{G}(j-1, j, j+1, l) = -(\delta_{j+1,l} - \delta_{j,l})(\vec{x}_{j+1} - \vec{x}_j) + (\delta_{j,l} - \delta_{j-1,l})(\vec{x}_j - \vec{x}_{j-1}) \quad (40)$$

then the gradient descent for (35) amounts to the following equation:

$$\dot{\vec{x}}_l = \sum_j F(j-1, j, j+1)\mathbf{G}(j-1, j, j+1, l). \quad (41)$$

Most of the terms in the (41) are zero. In fact, if we split

$$\begin{aligned} \mathbf{G}(j-1, j, j+1, l) &= -(\delta_{j+1,l} - \delta_{j,l})(\vec{x}_{j+1} - \vec{x}_j) + (\delta_{j,l} - \delta_{j-1,l})(\vec{x}_j - \vec{x}_{j-1}) & (42) \\ &= \mathbf{G}_1(j, j+1, l) + \mathbf{G}_2(j-1, j, l) & (43) \end{aligned}$$

then we see that $\mathbf{G}_1(j, j+1, l)$ is zero unless $j+1 \neq l$ and $j = l$ or $j+1 = l$ and $j \neq l$. Likewise $\mathbf{G}_2(j-1, j, l)$ is zero unless $j \neq l$ and $j-1 = l$ or $j = l$ and $j-1 \neq l$. Thus, the gradient descent (41) simplifies to:

$$\begin{aligned} \dot{\vec{x}}_l &= \sum_j F(j-1, j, j+1)\mathbf{G}_1(j, j+1, l) + \sum_j F(j-1, j, j+1)\mathbf{G}_2(j-1, j, l) & (44) \\ &= \sum_{j \neq l} F(j-1, j, l)\mathbf{G}_1(j, l, l) + F(l, j, j+1)\mathbf{G}_2(l, j, l). & (45) \end{aligned}$$

The above quantity reduces to:

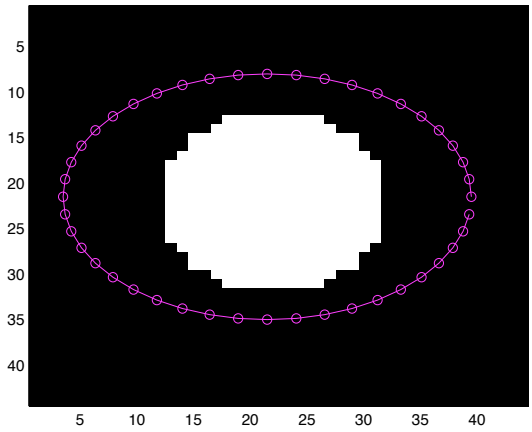
$$\dot{\vec{x}}_l = \sum_{j \neq l} \left(d_{j+1,j}^2 - 2d_{j,l}^2 + d_{j,j-1}^2\right)(\vec{x}_l - \vec{x}_j). \quad (46)$$

One needs to ensure that the above evolution takes place in the tangential direction to the curve. In the discrete setting, for each point on the curve $\vec{x}_j$, let us denote by: $\vec{T}_j = \vec{x}_{j+1} - \vec{x}_{j-1}$, the approximation to the tangent vector to the curve at $\vec{x}_j$. Thus, for each given evolved point: $\vec{x}_l$ get's mapped to it's projection onto the tangent vector $\vec{T}_l$. Thus $\vec{x}_l \rightarrow < \vec{x}_l, \vec{T}_l > \vec{T}_l$.

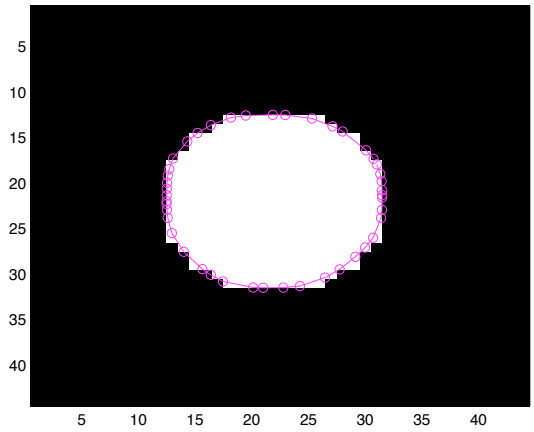
2.2 Proposed Model

Now, if f is a given image, $\{\mathbf{p}_i\}$ denote chosen grid points lying on the contour defining the shape of the evolving contour defined by ϕ and $\{\mathbf{r}_i\}$ are those lying on the reference contour ψ. Then the proposed shape prior active contour model is the following:

$$\inf_{\phi} \left\{ E(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N) = \sum_{i,j} \left(\|\mathbf{p}_i - \mathbf{p}_j\|^2 - \|\mathbf{r}_i - \mathbf{r}_j\|^2\right)^2 + \int_0^1 g(x(p), y(p))|\mathcal{C}'(p)|dp \right\}. \quad (47)$$

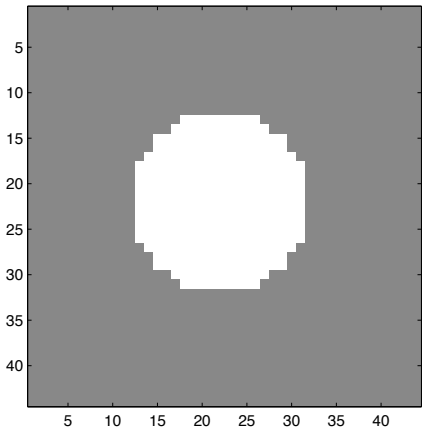


Given image (background) and initial (purple)

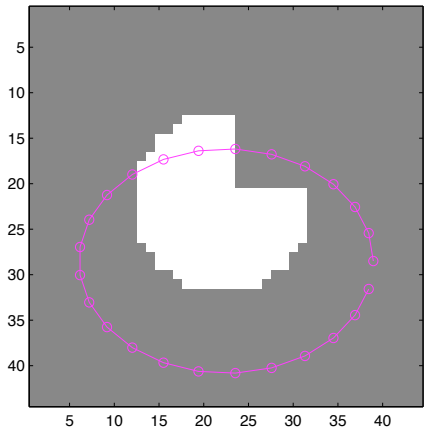


initial (blue), evolved (purple)

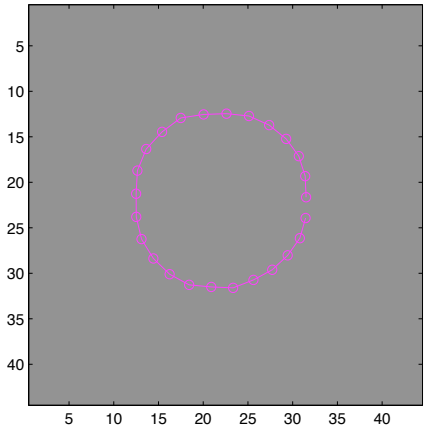
Figure 3: *Snakes:*



Given image

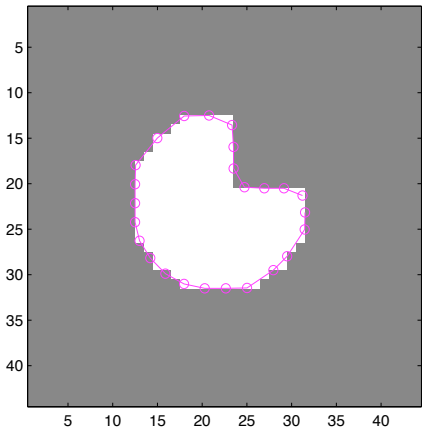


Occluded image, initial contour (purple)

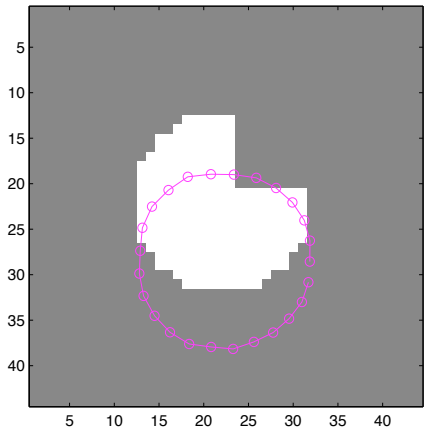


Prior

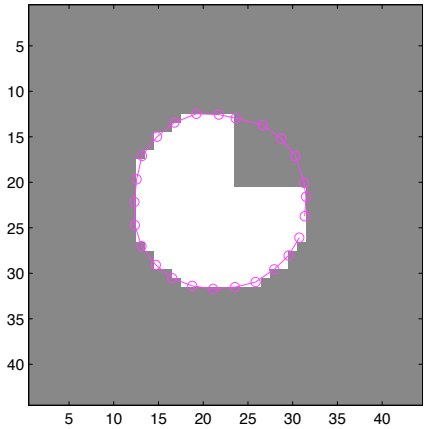
Figure 4: *Snakes:*



Snakes (GAC)

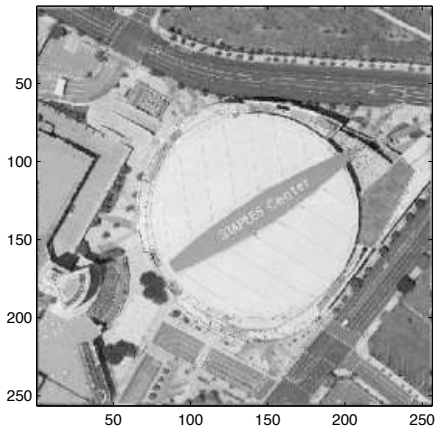


Cliques

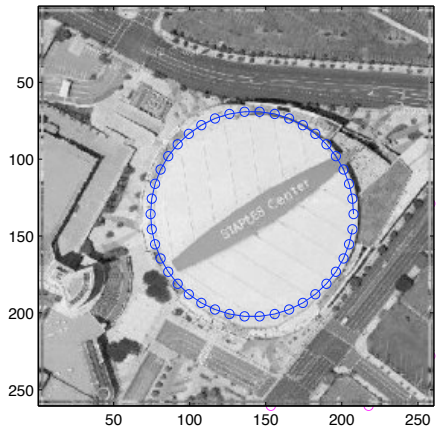


Proposed Model

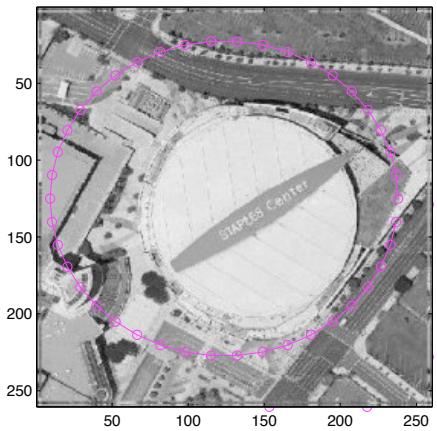
Figure 5: *Snakes:*



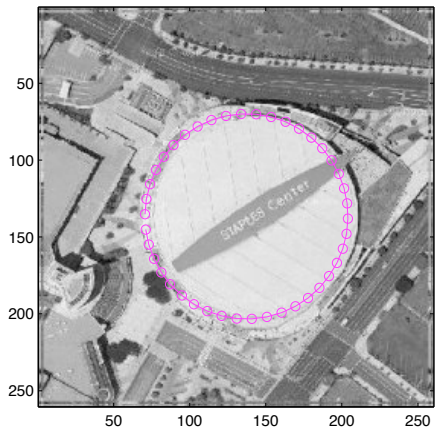
Staples Center Image (background)



Shape Prior (blue)

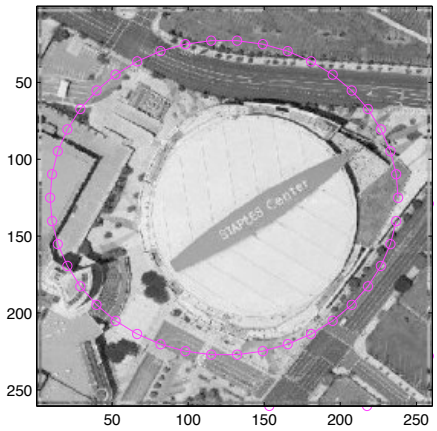


Initial Contour (purple)

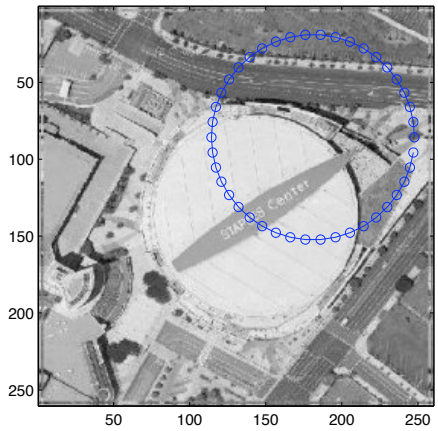


Evolved Contour (purple) Proposed Model

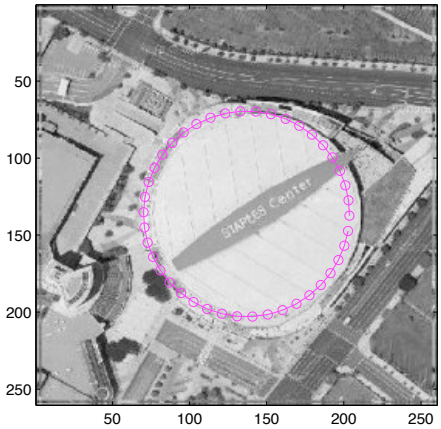
Figure 6: *Staples Center Segmentation:*



Staples Center Image (background)

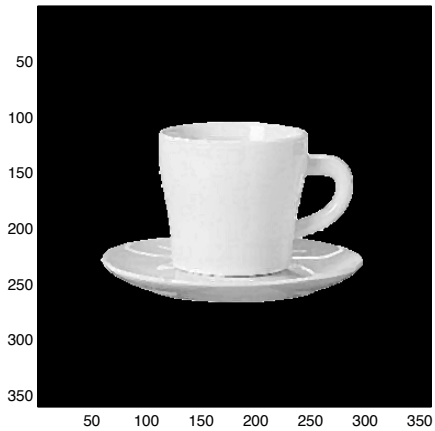


Shape Prior (blue)

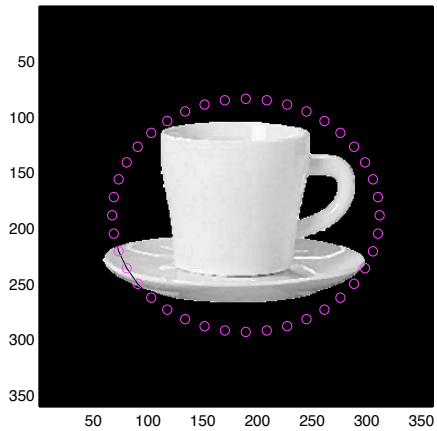


Evolved Contour (purple) from Proposed Model

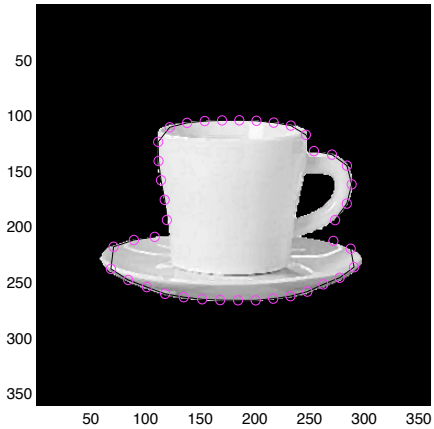
Figure 7: *Staples Center Segmentation:*



initial image

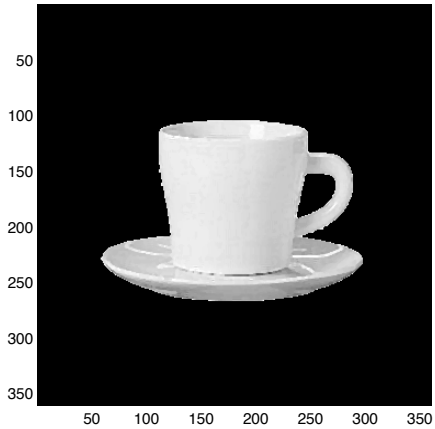


initial contour

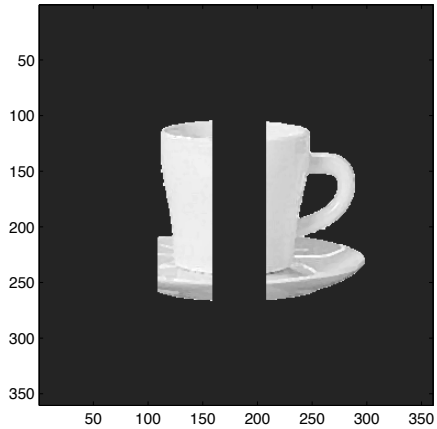


Evolved Contour (Snakes, no prior)

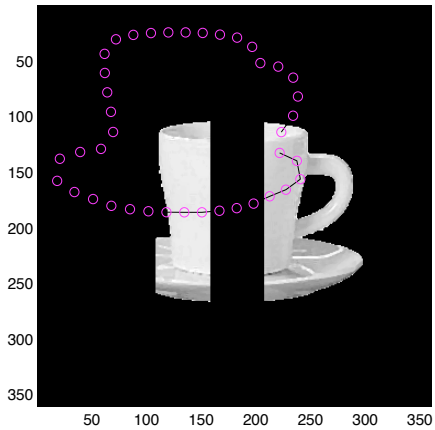
Figure 8: *Snakes Segmentation, No Shape Prior:*



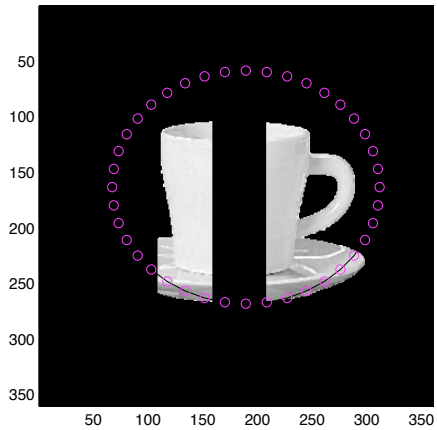
clean image



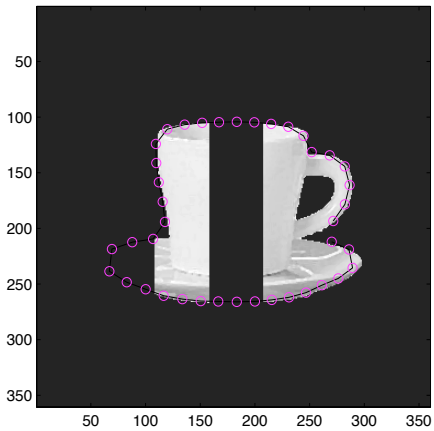
occluded image



prior

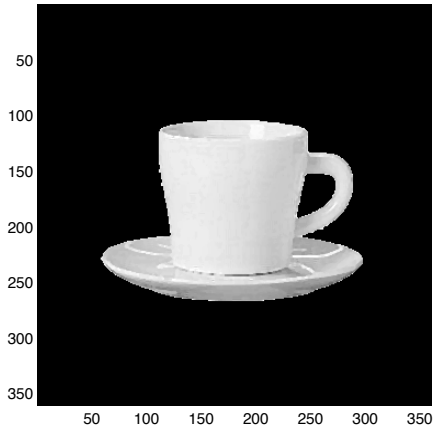


initial contour (purple)

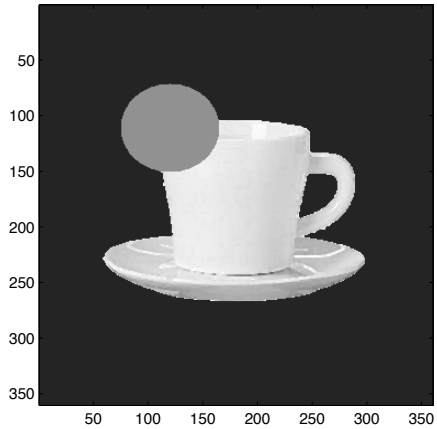


Evolved Contour (Proposed Model)

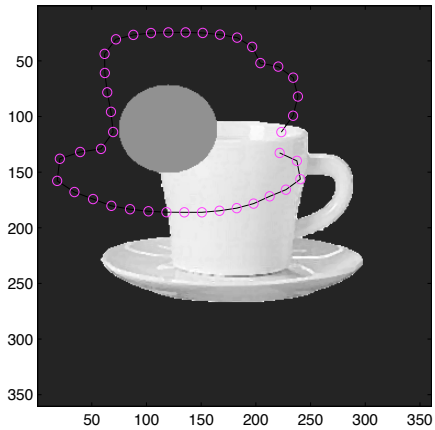
Figure 9: *Shape Prior Segmentation:*



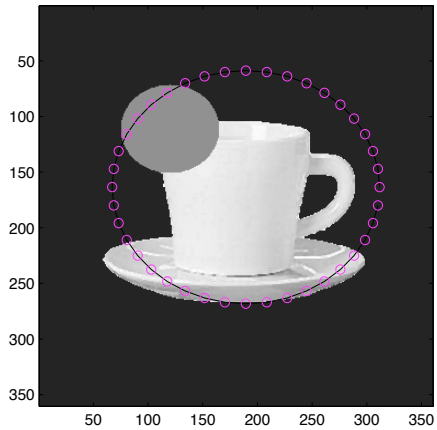
clean image)



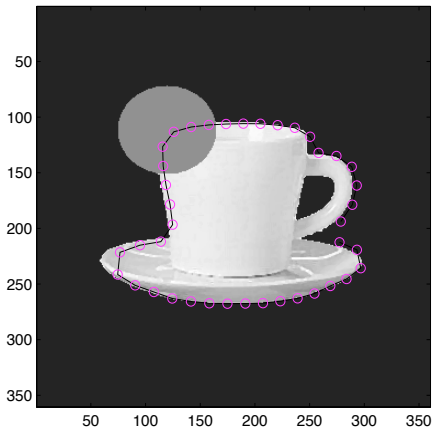
occluded image



prior



initial contour (purple)



Evolved Contour (Proposed Model)

Figure 10: *Shape Prior Segmentation:*