Texture modeling using total variation minimization and oscillating patterns in image processing

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References

Total Variation Minimization

(Rudin-Osher-Fatemi)

Oscillating Patterns

(Y. Meyer)

Image decomposition models into cartoon + texture:

1. *Modeling textures with total variation minimization and oscillating patterns in image processing*, L. Vese and S. Osher, May 2002 (to appear in JSC)

2. *Image decomposition and restoration using total variation and the* H^{-1} *norm,* S. Osher, A. Sol and L. Vese, October 2002 (to appear in SIAM)

(talk by S. Osher)

Let $f : R^2 \to R$ be a given image, $f \in L^2(R^2)$

Standard linear degradation model: $f = u + v$

$u =$ true original image

$v =$ additive noise or texture, such that $\int v(x, y)dxdy = 0$

Total variation minimization for image restoration (Rudin-Osher-Fatemi, 1992)

$$\inf_{u \in BV} F(u) = |u|_{BV} + \lambda \| f - u \|_{L^2}^2,$$ with $\lambda > 0$ a parameter,

or

$$\inf_{u \in BV} \left\{ F(u) = |u|_{BV} + \lambda \| v \|_{L^2}^2, \quad f = u + v \right\}$$

BV is the space of functions of bounded variation

$|u|_{BV}$ is the associated semi-norm, i.e. the total variation of u

Definition of *BV* as a dual space:

$u \in BV$ if $u \in L^1$ and

$|u|_{BV} :=$

$$\int |\nabla u| = \sup_{\vec{g}} \left\{ \int u(x,y)\, \mathrm{div}\vec{g}(x,y)\,dxdy : \ \vec{g} \in C_0(R^2;R^2),\ \|\vec{g}\|_{L^\infty} \le 1 \right\} < \infty$$

In general:

If X and X^* are dual spaces, then $\|u\|_X = \sup_{v \in X, \|v\|_{X^*} \le 1} \int uv.$

Taking $X = BV$, then $X^* = BV^* = \{ f = \mathrm{div}\,\vec{g} \mid \vec{g} \in C_0 \}$,

with $\|f\|_{X^*} = \inf_{\vec{g},\, f = \mathrm{div}\,\vec{g}} \|\vec{g}\|_{L^\infty}$

The associated Euler-Lagrange equation:

$$\inf_u F(u) \ \Rightarrow \ u = -\frac{1}{2\lambda} div\left(\frac{\nabla u}{|\nabla u|} \right) + f \quad \text{(formally)}$$

Properties of the ROF model

- It is an energy functional with linear growth
- There exists a unique minimizer $u \in BV$
- The BV space allows for discontinuities along curves, therefore edges in the image
- The model denoises well images, while keeping sharp edges
- It can be easily solved in practice by finite differences :

$$\inf_u F(u) \;\Rightarrow\; u = -\frac{1}{2\lambda} div\left(\frac{\nabla u}{|\nabla u|}\right) + f \quad \text{(formally)}$$

- Only a "*cartoon*" approximation u of f is kept. The residual $v := f - u$ (texture + noise) is removed from f.
- The model should decompose a given image f into a cartoon part u and an oscillatory part v (noise / texture).
- However, this is not always the case. For example :

Limitations, remarks and motivations for a new model:

Let $f = \alpha\chi_D$, with $\alpha =$ constant and χ_D the characteristic function of a disk D of radius R.

Then :

If $\lambda R \geq \frac{1}{\alpha}$: the minimizer is $u = (\alpha - \frac{1}{\lambda R})\chi_D$, and

the residual is $v := f - u = \frac{1}{\lambda R}\chi_D$

Therefore, $u \neq f$ and v is not an oscillatory function.

In conclusion, the model does not always produce a decomposition into cartoon + texture.

Given $f \in L^2$, let u be the unique minimizer of $F(u)$.

Then the residual $v := f - u$ is given by

$$v := f - u = -\frac{1}{2\lambda} div\left(\frac{\nabla u}{|\nabla u|}\right)$$

or

$$v = \frac{\partial}{\partial x}\left(-\frac{u_x}{2\lambda|\nabla u|}\right) + \frac{\partial}{\partial y}\left(-\frac{u_y}{2\lambda|\nabla u|}\right)$$

Then, if u is a minimizer, the residual $v=f-u$ is given by:

$$v := f - u = \mathrm{div}\vec{g}, \text{ with } \vec{g} \in L^\infty, \text{ s.t. } \|\vec{g}\|_{L^\infty} = \frac{1}{2\lambda} < \infty$$

Therefore: the residual v could be characterized by another norm, instead of the L^2 norm.

Y. Meyer proposes a better norm for the oscillatory component v, instead of $\| v \|_{L^2}^2$: $v \in G$ = dual space of $W^{1,1}$ (larger than BV^*)

Def. Let G = Banach space of generalized functions

$v(x,y) = \partial_x g_1(x,y) + \partial_y g_2(x,y) = div(\vec{g}), \quad g_1, g_2 \in L^\infty$

induced with the norm

$$\|v\|_* = \inf_{g_1, g_2 \in L^\infty,\ v = div(\vec{g})} \left\| \sqrt{g_1^{\ 2} + g_2^{\ 2}} \right\|_{L^\infty}$$

Y. Meyer proposes a new image decomposition model:

$$\inf_{u \in BV} \left\{ E(u) = | u |_{BV} + \lambda \| v \|_*, \ f = u + v \right\}$$

Remark: we have the inclusions

$$W^{1,1} \subset BV \subset L^2 \subset G = (W^{1,1})^*$$

$$BV^* \subset G$$

Remarks:

- Textures are represented by 2 functions only (other techniques, based on Gabor transform, are much more expensive)

- The *-norm distinguishes between different textures:

<u>Example in 1D:</u>

$$f_1(x) = \cos(\alpha x) \qquad f_2(x) = \cos(\beta x), \qquad \alpha, \beta \neq 0$$

$$\| f_1 \|_* = \frac{1}{\alpha} \qquad \neq \qquad \| f_2 \|_* = \frac{1}{\beta} \qquad \text{if} \qquad \alpha \neq \beta$$

$\Rightarrow$ the $*-$norm can distinguish between different textures

- Oscillating patterns have small norms in G.

If $f_n(x) = \cos(nx)$, with $n \to \infty$ (more oscillations), then

$$\| f_n \|_* = \frac{1}{n} \to 0.$$

Remark: the model of Y. Meyer cannot be directly solved in practice

Our contribution: to propose a simple practical algorithm for solving this new model and exploit its applications

- image decomposition into cartoon + texture
- texture segmentation
- image restoration

Using: $$\left\| \sqrt{g_1^{\,2} + g_2^{\,2}} \right\|_{L^\infty} = \lim_{p \to \infty} \left\| \sqrt{g_1^{\,2} + g_2^{\,2}} \right\|_{L^p}$$

The proposed new (convex) minimization model is:

For large parameters $\lambda, p,$ solve:

$$\operatorname*{Inf}_{u\in BV,\vec{g}\in L^p} F(u,\vec{g}) = |u|_{BV} + \lambda \| f - (u + \operatorname{div}\vec{g}) \|_{L^2}^2 + \mu \| \vec{g} \|_{L^p}$$

or

$$\operatorname*{Inf}_{u\in BV,\vec{g}\in L^p} F(u,\vec{g}) = \int |\nabla u| + \lambda \int | f - (u + \operatorname{div}\vec{g}) |^2 \, dxdy$$

$$+ \mu \left(\int \sqrt{g_1^{\,2} + g_2^{\,2}}^{\,p} dxdy \right)^{1/p}$$

If $\lambda \to \infty$ and $p \to \infty,$ this model approximates the model proposed by Y. Meyer:

- First term insures that $u \in BV$
- Second term insures that $f \cong u + \operatorname{div}(\vec{g}) = u + v$
- Third term is a penalty on $\|v\|_*$

The Euler-Lagrange equations (now easily obtained)

Case *p*>>1 (global terms)

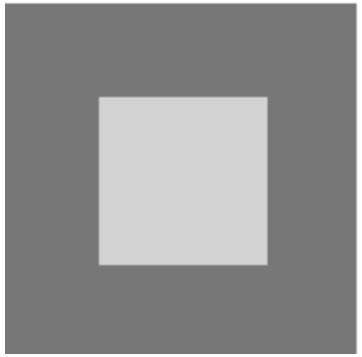
$$\underset{u\in BV,\vec{g}\in L^p}{Inf}\ F(u,\vec{g}) \Rightarrow \begin{cases} u = f - \mathrm{div}\vec{g} + \dfrac{1}{2\lambda}\mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) \\ \mu\|\vec{g}\|_p^{\,1-p}\,|\vec{g}|^{p-2}\,g_1 = 2\lambda\left(\dfrac{\partial}{\partial x}(u-f) + \partial^2{}_{xx}g_1 + \partial^2{}_{xy}g_2\right) \\ \mu\|\vec{g}\|_p^{\,1-p}\,|\vec{g}|^{p-2}\,g_2 = 2\lambda\left(\dfrac{\partial}{\partial y}(u-f) + \partial^2{}_{xy}g_1 + \partial^2{}_{yy}g_2\right) \end{cases}$$

Case *p*=1 (local terms only, faster, but also gives good results)

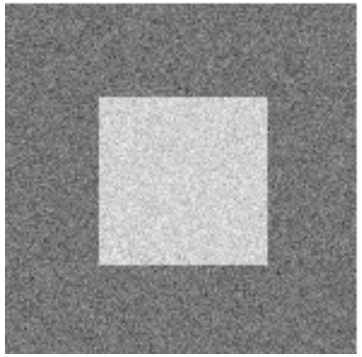
$$\underset{u\in BV,\vec{g}\in L^1}{Inf}\ F(u,\vec{g}) \Rightarrow \begin{cases} u = f - \mathrm{div}\vec{g} + \dfrac{1}{2\lambda}\mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) \\ \mu\dfrac{g_1}{|\vec{g}|} = 2\lambda\left(\dfrac{\partial}{\partial x}(u-f) + \partial^2{}_{xx}g_1 + \partial^2{}_{xy}g_2\right) \\ \mu\dfrac{g_2}{|\vec{g}|} = 2\lambda\left(\dfrac{\partial}{\partial y}(u-f) + \partial^2{}_{xy}g_1 + \partial^2{}_{yy}g_2\right) \end{cases}$$

Image denoising and comparisons with ROF model

Original



Noisy



Original

Noisy

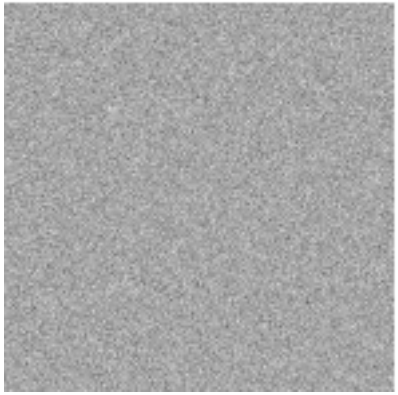
New model u

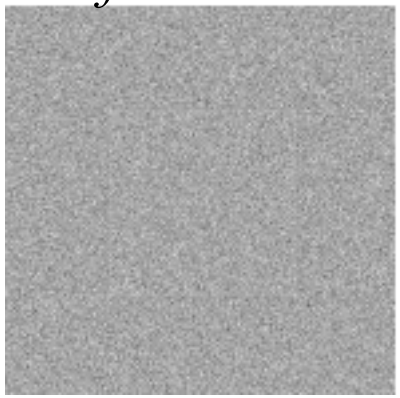
ROF model u

New model u

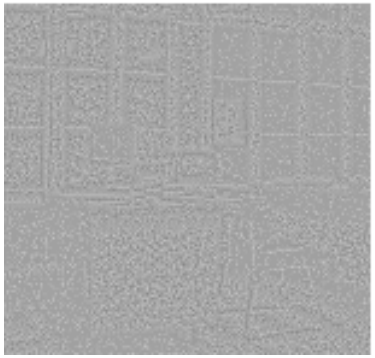
ROF model u

Residuals $v=f-u$



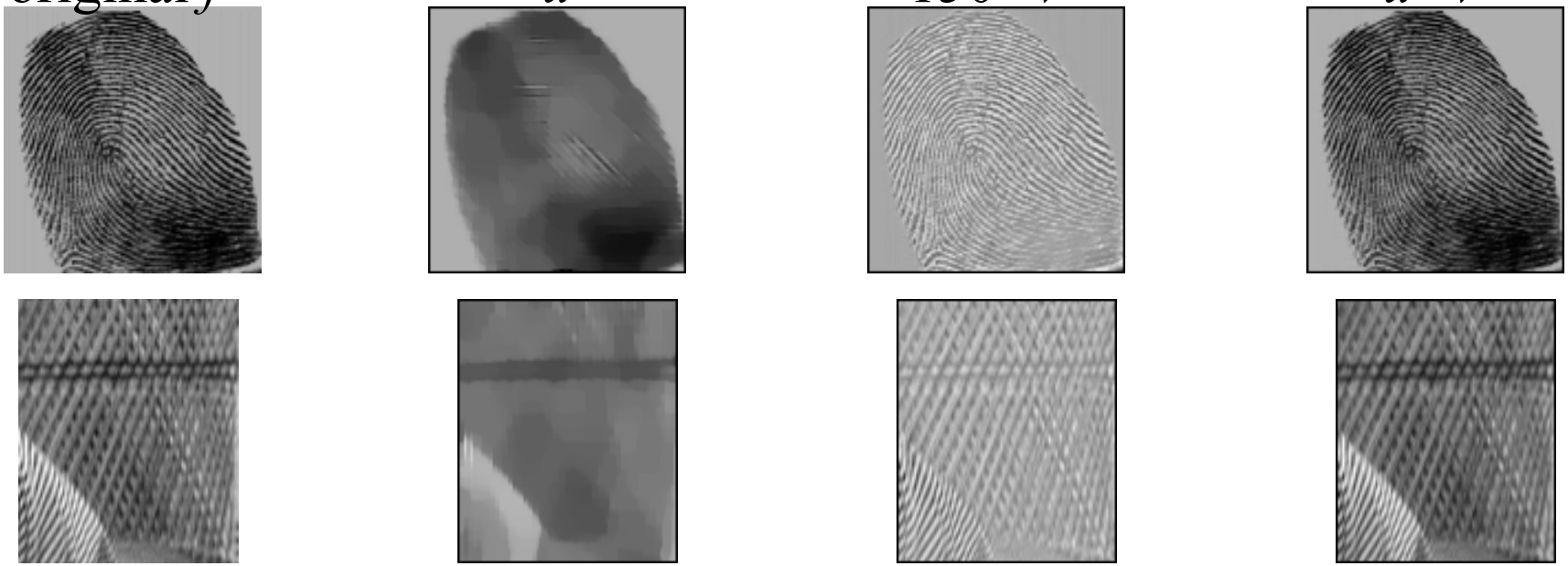


Residuals $v=f-u$

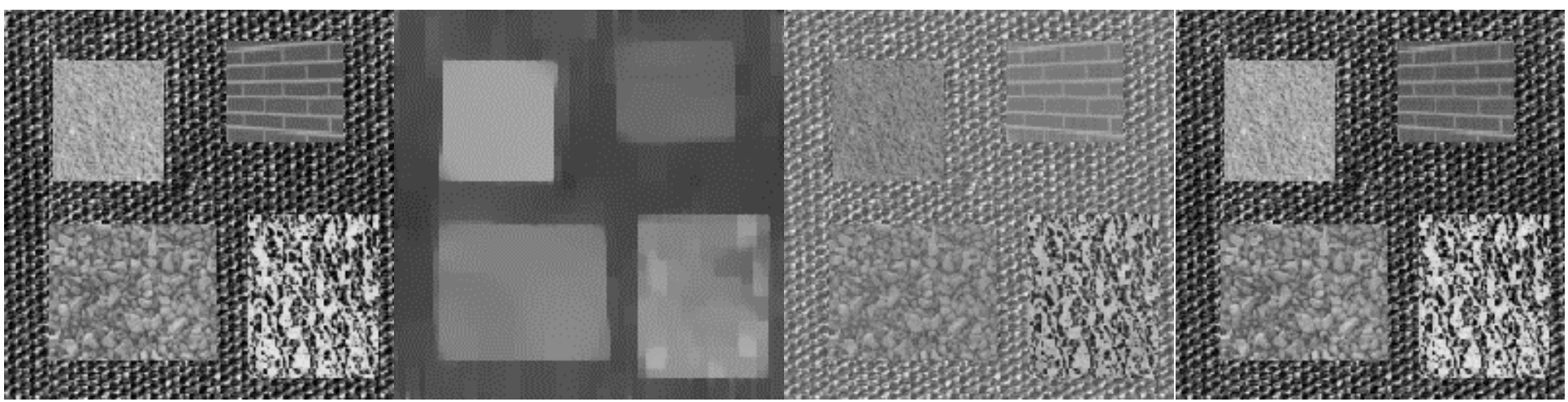


Cartoon + Texture image decomposition

original f | u | $150+v$ | $u+v$



original $=f$ | u | $150+v$ | $u+v$



f

$u+v$

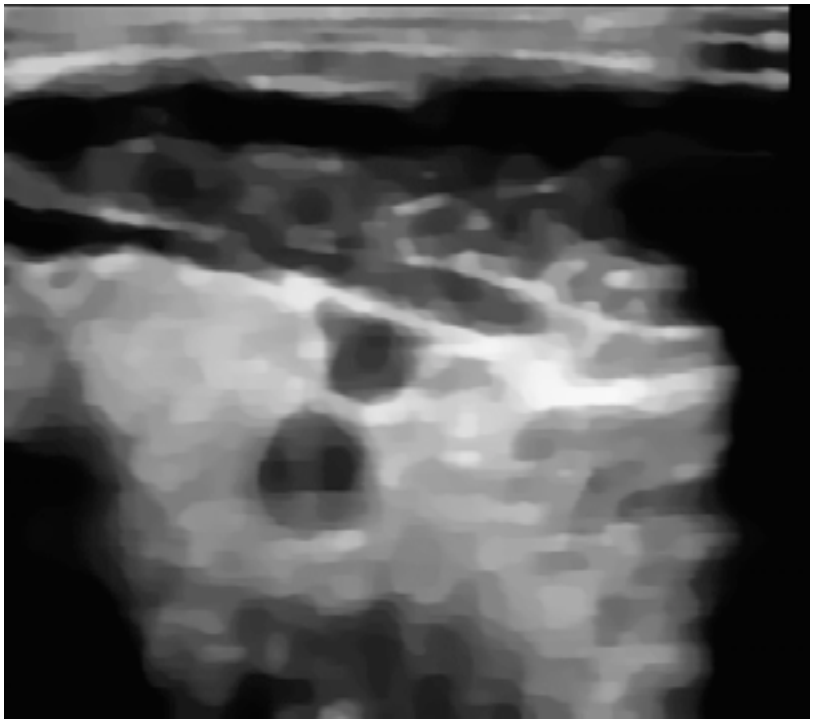
u

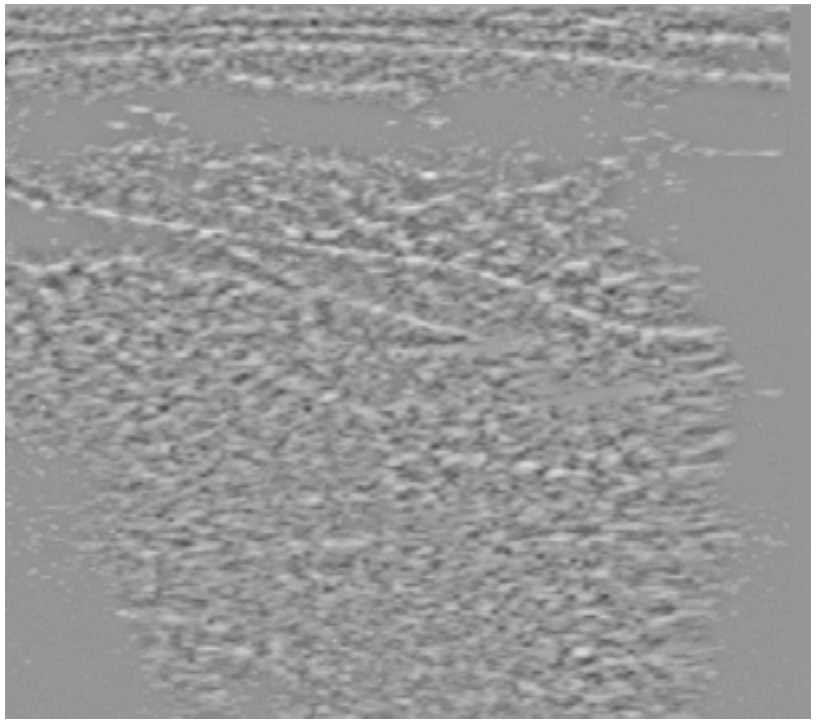
v

Ultrasound image decomposition

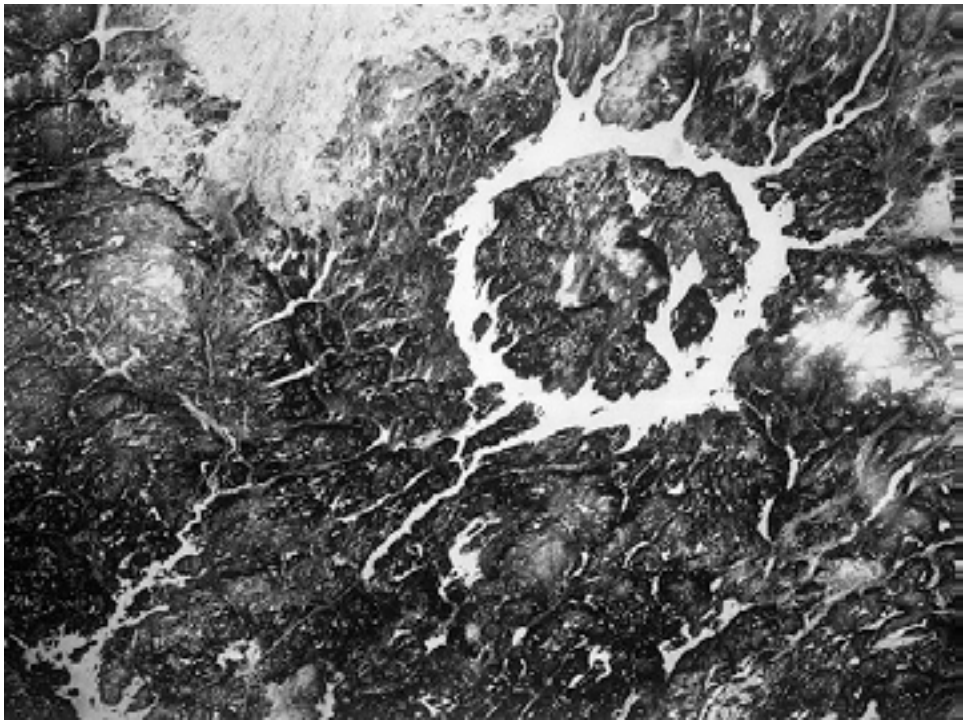
$f=u+v$

u



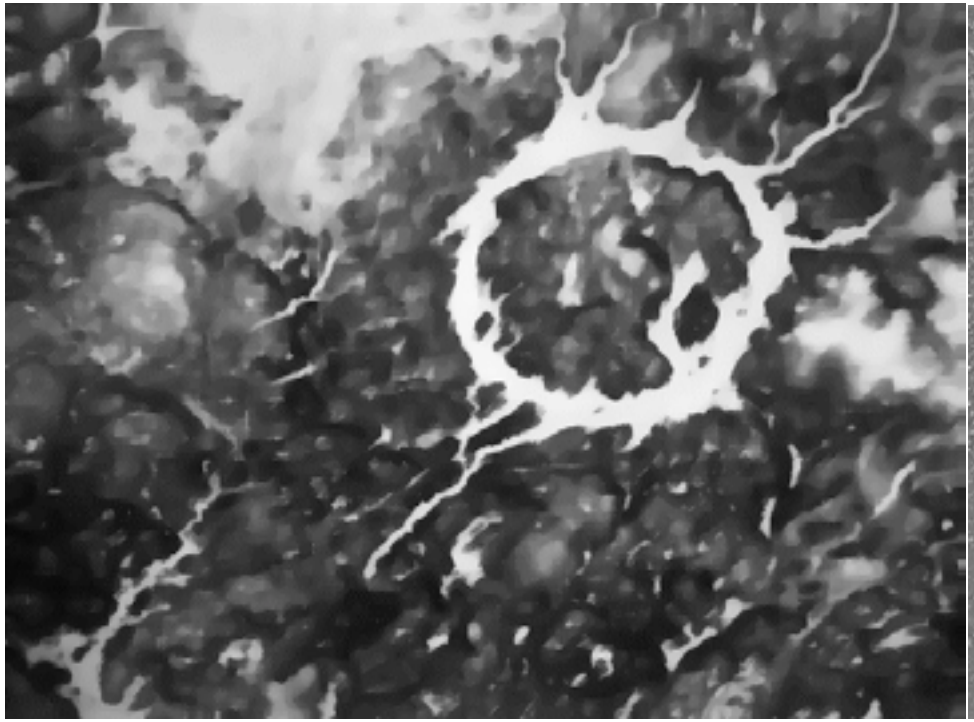


v



f

u



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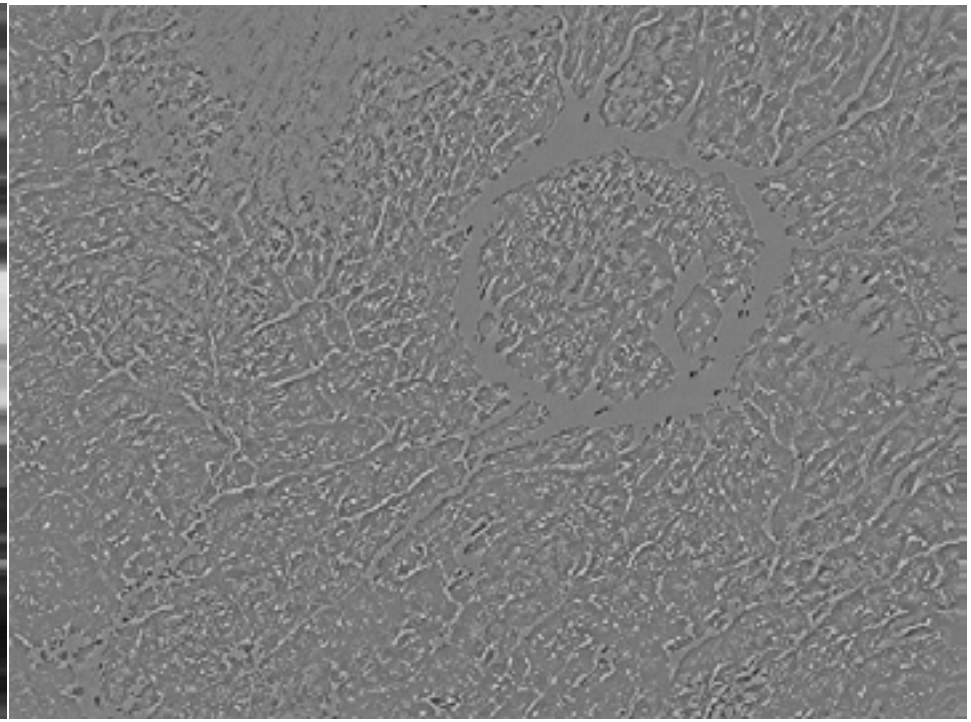
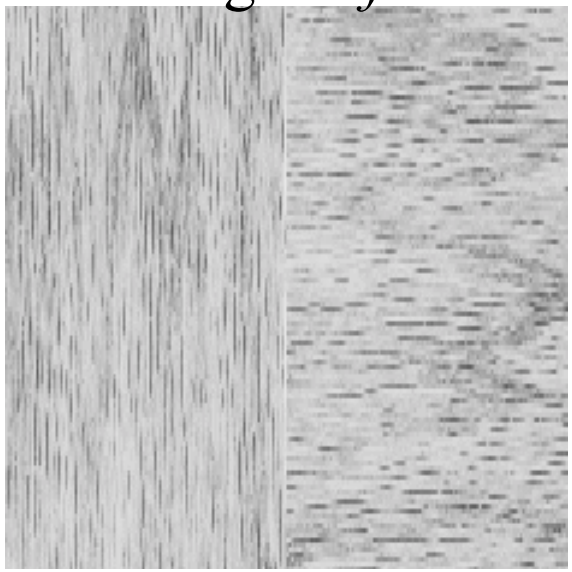
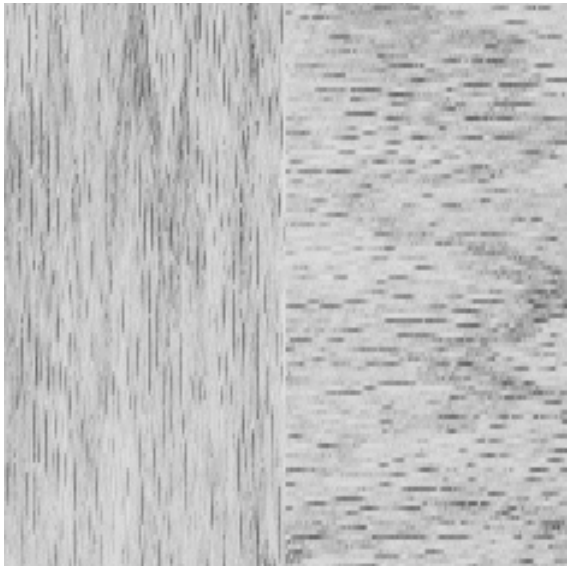


original f



$u+v$



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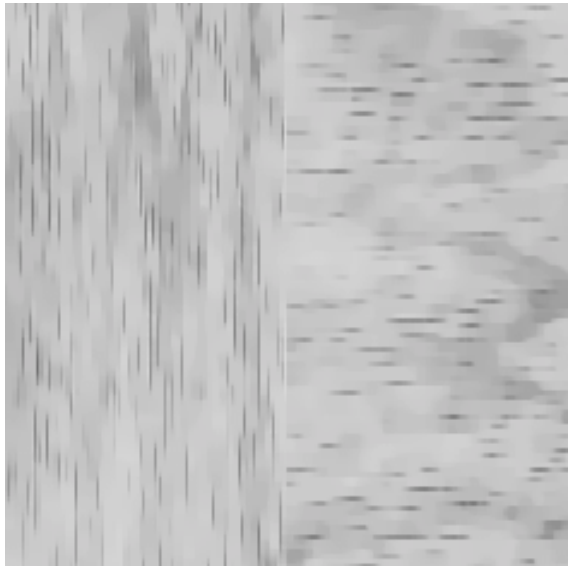
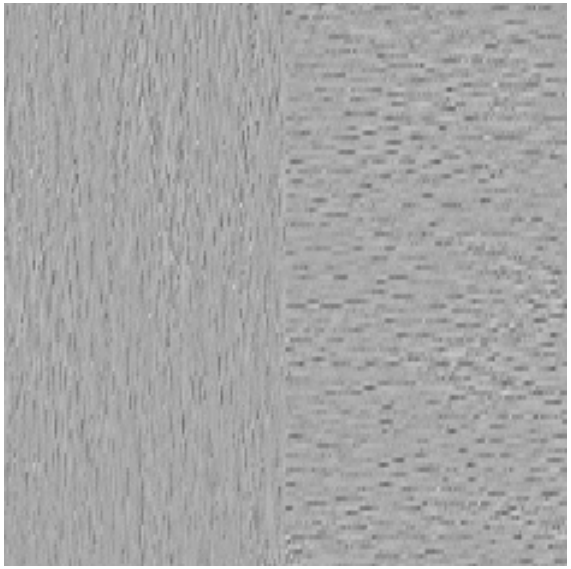
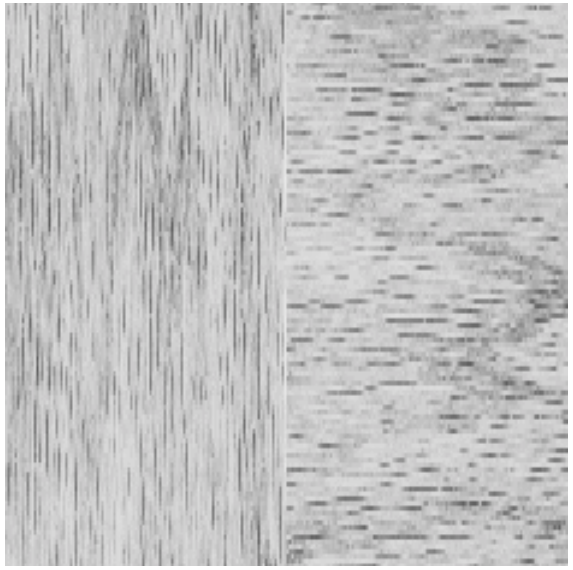


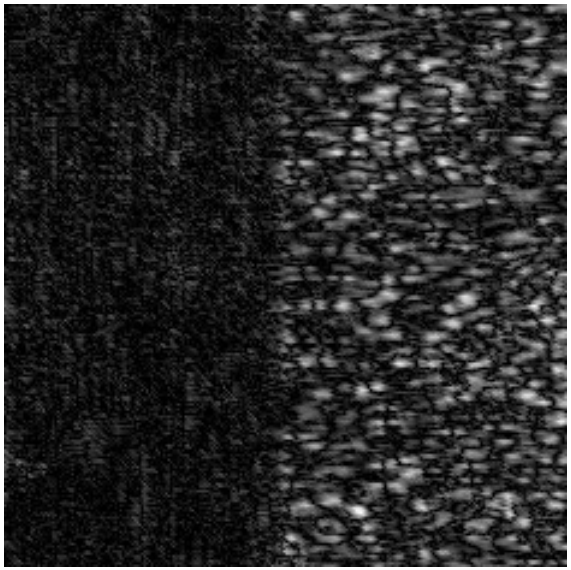
$150+v$



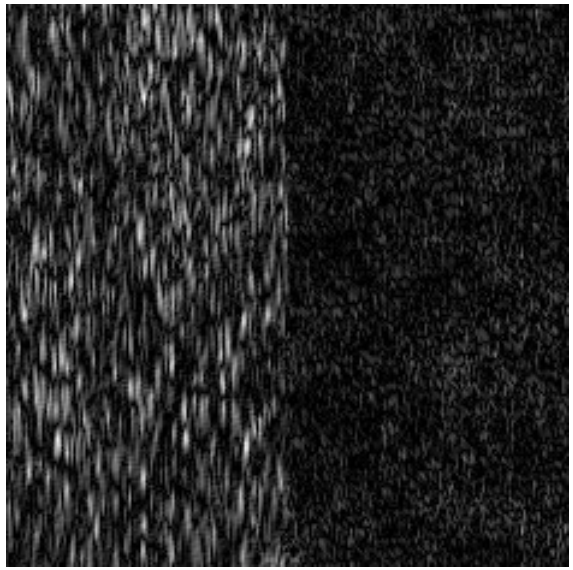
original



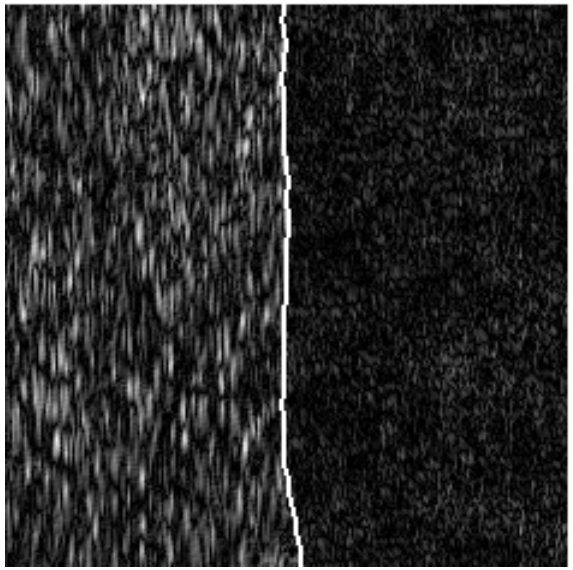
$|g_1|$



$|g_2|$



Texture segmentation



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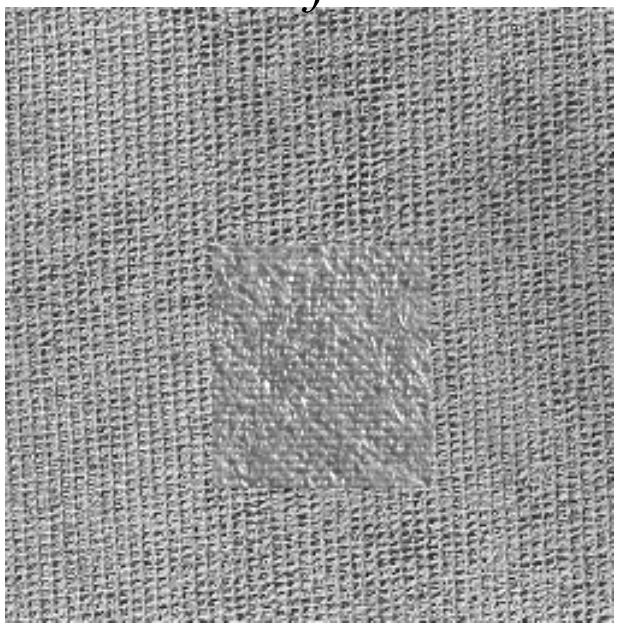

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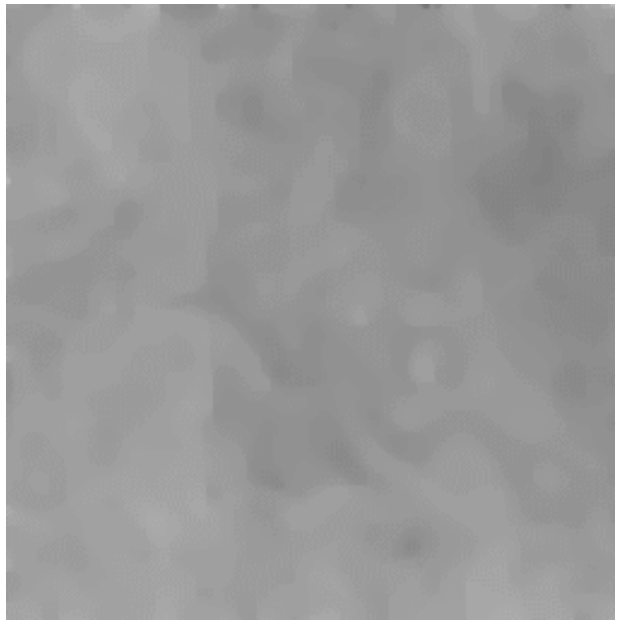
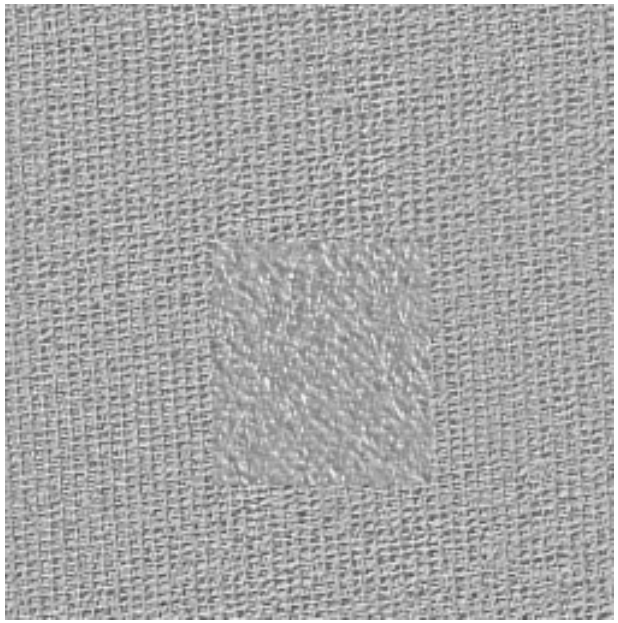
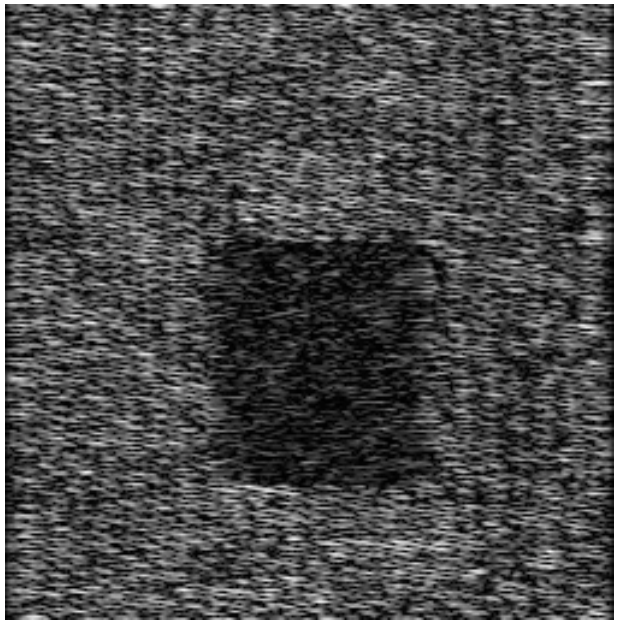


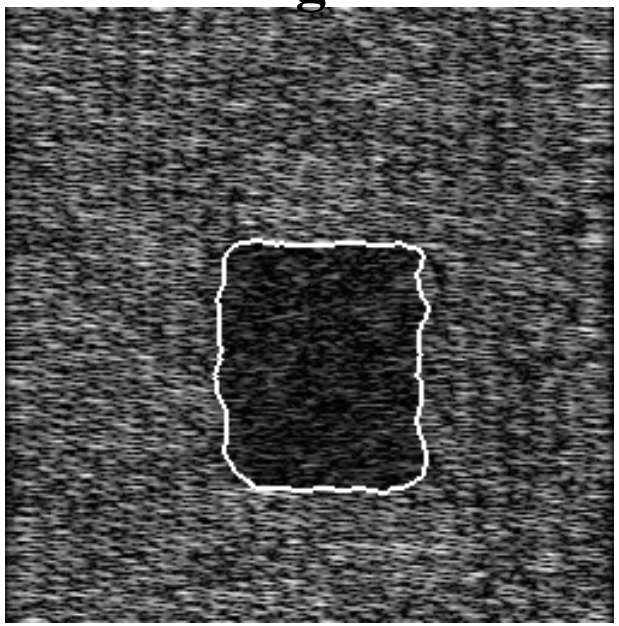
v



$|g_2|$



Texture segmentation



Contour detected using the active contour model without edges (Chan-Vese)