Image Segmentation using Level Sets and Energy Minimization Techniques

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References

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PLAN

I. Active contours without edges

II. Generalization to the Mumford-Shah segmentation model

- The piecewise-constant case
- The piecewise-smooth case

III. Anisotropic energies:

Total Variation-based active contours using level sets

Anisotropic Mumford-Shah like models

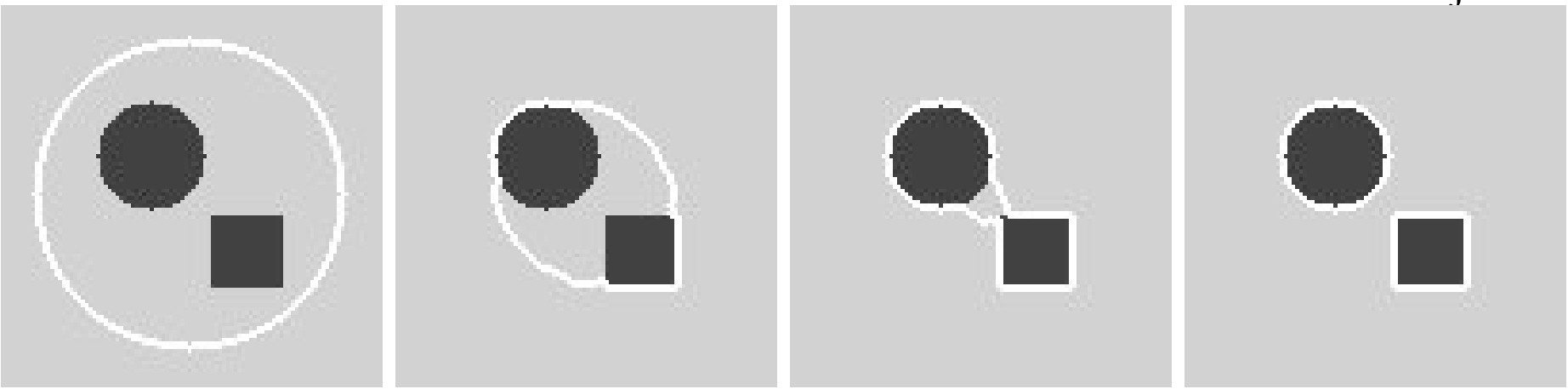
What is an active contour ?

Giving an image $u_0 : \Omega \rightarrow \Re$

Evolve a curve C to detect objects in u_0

The curve has to stop on boundaries of objects

Initial Curve → Evolutions → Detected Objects



Basic idea in classical active contours

Curve evolution & deformation (internal forces):

Ex: Min *Length(C)*

Boundary detection (external forces):

Stopping edge-function $g(|\nabla u_0|)$

$$g(|\nabla u_0|) = \frac{1}{1 + |\nabla G_\sigma * u_0|^p}$$

Some active contours with edge-function

***Snake model*:**

Kass, Witkin, Terzopoulos 88

***Geometric model*:**

Caselles, Catte, Coll, Dibos 93

Malladi, Sethian, Vemuri 93

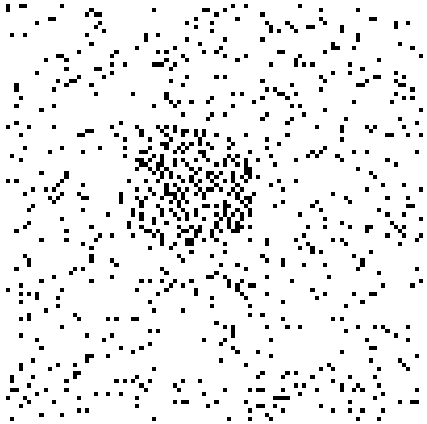
***Geodesic model*:**

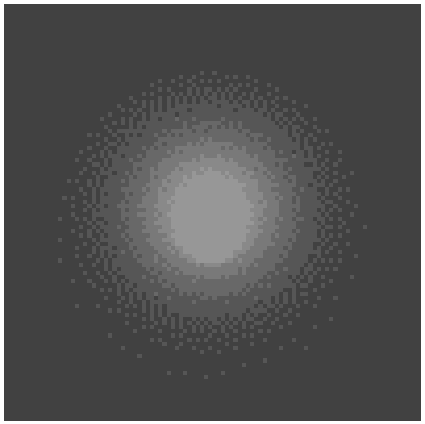
Caselles, Kimmel, Sapiro 95

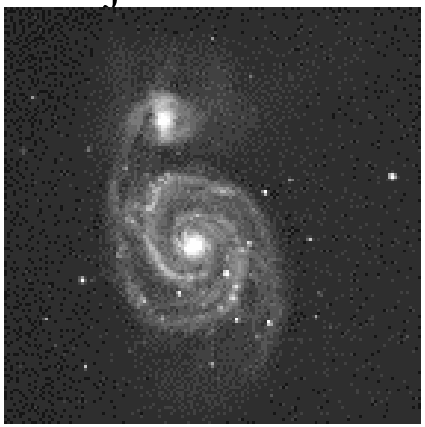
Kichenassamy, Kumar, Olver, Tannenbaum, Yezzi 95

Limitations

- detect objects with edges defined by gradient only
- interior contours are not automatically detected
- the initial curve has to surround the objects







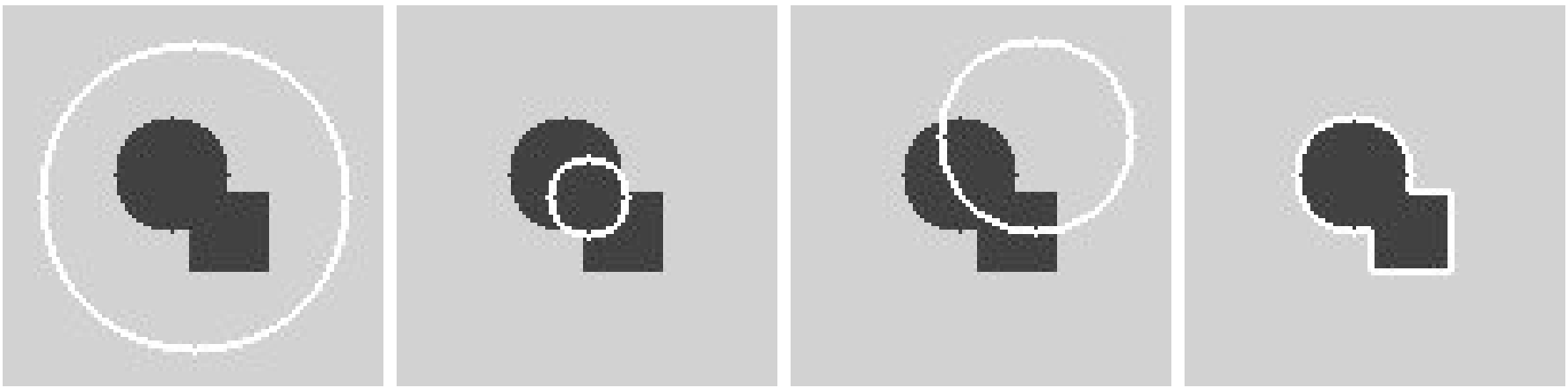
A fitting term without edges

$$\int_{inside(C)} |u_0 - c_1|^2 dxdy + \int_{outside(C)} |u_0 - c_2|^2 dxdy$$

$c_1 = average(u_0)$ inside C

$c_2 = average(u_0)$ outside C

Fit > 0 | Fit > 0 | Fit > 0 | Fit ~ 0



An active contour model without edges

$$\inf_{c_1,c_2,C} F(c_1,c_2,C) = \mu Length(C) + \lambda \int_{inside(C)} |u_0 - c_1|^2 dxdy + \lambda \int_{outside(C)} |u_0 - c_2|^2 dxdy$$

Fitting not depending on gradient: detects contours without gradient

Relation with Mumford-Shah segmentation model 89

$$\inf_{u,C} F^{MS}(u,C) = \mu \cdot |C| + \lambda \int_{\Omega} |u - u_0|^2 \, dxdy + \int_{\Omega - C} |\nabla u|^2 \, dxdy$$

u is an optimal approximation of the initial image

C is the set of jumps or edges of u

u is smooth outside the edges C

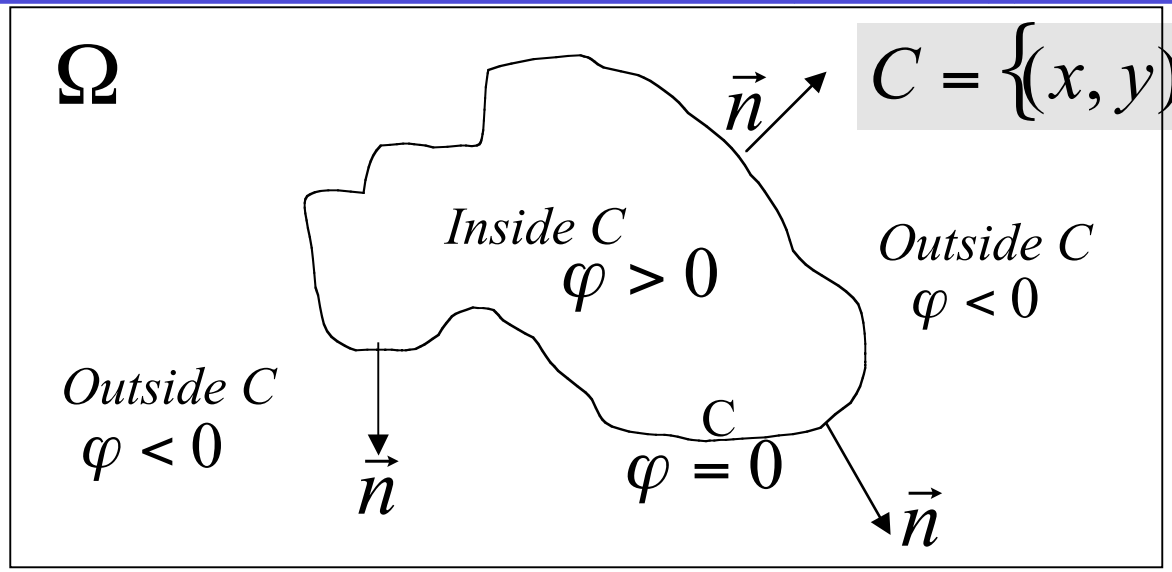
The model is a particular case of the Mumford-Shah functional

$$u = \begin{cases} c_1 = \text{average}(u_0) & \text{inside} \quad C \\ c_2 = \text{average}(u_0) & \text{outside} \quad C \end{cases}$$

u is the best approximation of u_0 taking only two values c_1 and c_2, and with one edge C

the snake C is the boundary between the sets $\{u = c_1\}$ and $\{u = c_2\}$

Level Set Representation (Osher-Sethian 87)



$$C = \{(x, y) \mid \varphi(x, y) = 0\}$$

Allows automatic topology changes (merging, breaking), cusps,

Variational Formulations and Level Sets

(following Zhao, Chan, Merriman, Osher 96, Evans-Gariepy)

The Heaviside function:

$$H(\varphi) = \begin{cases} 1, \text{if } \varphi \geq 0 \\ 0, \text{if } \varphi < 0 \end{cases}$$

Length or perimeter:

$$|C| = \int_{\Omega} |\nabla H(\varphi)|$$

The level set formulation of the active contour model

$$\inf_{c_1,c_2,\varphi} F(c_1,c_2,\varphi) = \mu\int_\Omega |\nabla H(\varphi)|$$

$$+\lambda\int_\Omega |u_0(x,y)-c_1|^2 H(\varphi)dxdy + \lambda\int_\Omega |u_0(x,y)-c_2|^2 (1-H(\varphi))dxdy$$

$$u(x,y) = c_1 H(\varphi(x,y)) + c_2(1-H(\varphi(x,y)))$$

The Euler-Lagrange equations

Equations for c_1 and c_2

$$c_1(t) = \frac{\int_\Omega u_0 H(\varphi)dxdy}{\int_\Omega H(\varphi)dxdy}, \quad c_2(t) = \frac{\int_\Omega u_0(1-H(\varphi))dxdy}{\int_\Omega (1-H(\varphi))dxdy}$$

Equation for $\varphi = \varphi(t,x,y)$

$$\frac{\partial\varphi}{\partial t} = \delta_\varepsilon(\varphi)\left[\mu\cdot div\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) - \lambda(u_0-c_1)^2 + \lambda(u_0-c_2)^2\right]$$

$$\varphi(0,x,y) = \varphi_0(x,y)$$

Existence of minimizers

(among characteristic functions of sets with finite perimeter)

$$F(c_1(\varphi), c_2(\varphi), \varphi) = \overline{F}(H(\varphi)), \qquad H(\varphi) = \aleph_{\{\varphi > 0\}}$$

$$\overline{F}(\aleph) = \mu \int_\Omega |\nabla \aleph| + \lambda \int_\Omega |u_0 - c_1(\aleph)|^2 \, \aleph(x) dx + \lambda \int_\Omega |u_0 - c_2(\aleph)|^2 \, (1 - \aleph(x)) dx$$

Theorem: Let $\Omega \subset R^N$ be open, bounded with $\partial\Omega$ Lipschitz.

If $u_0 \in L^\infty(\Omega)$, then the following minimization problem

$$\inf_{\aleph} \overline{F}(\aleph), \quad \aleph \in BV(\Omega), \quad \aleph(x) \in \{0,1\} dx - a.e.,$$

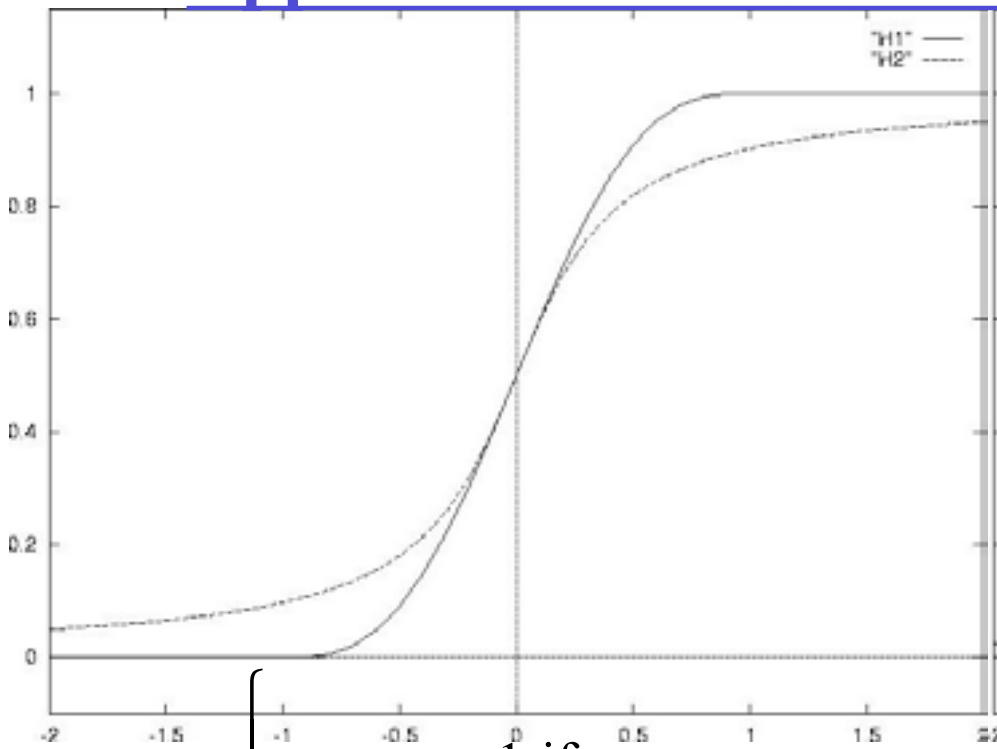
has a solution.

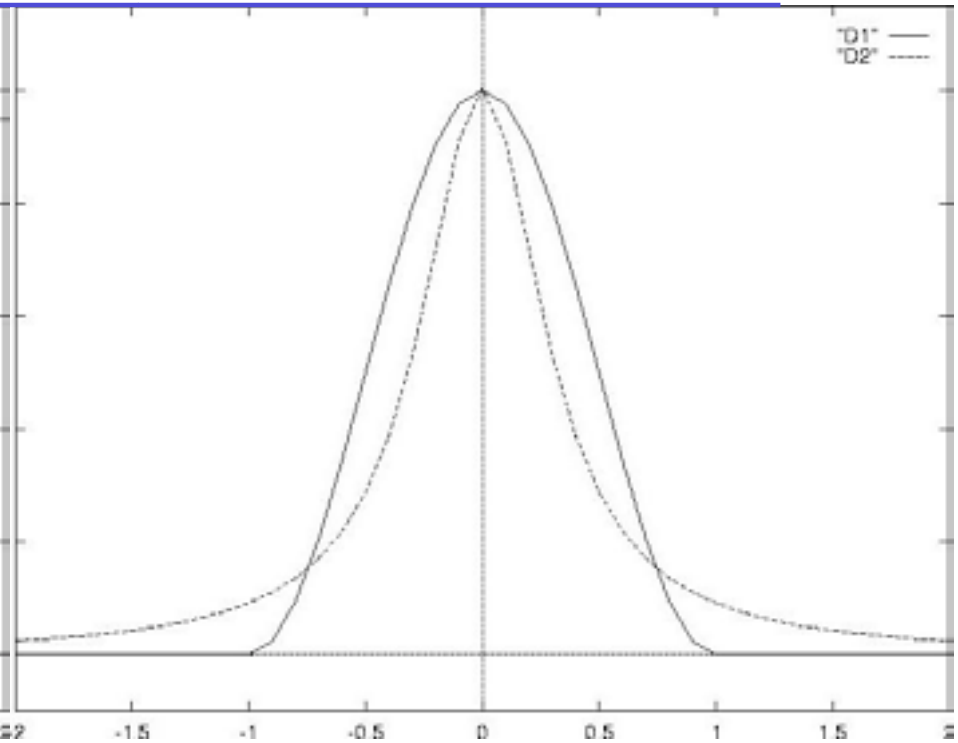
$BV(\Omega)$ = space of functions of bounded variation

Proof : standard in calculus of variations

(*lsc* of TV and compactness in BV)

Approximations of Heaviside and Delta functions





$$H_\varepsilon(x) = \begin{cases} 1, \text{if } x > \varepsilon \\ 0, \text{if } x < -\varepsilon \\ \frac{1}{2}\left[1 + \frac{x}{\varepsilon} + \frac{1}{\pi}\sin\left(\frac{\pi x}{\varepsilon}\right)\right], \text{if } |x| \le \varepsilon \end{cases}$$

$$H_\varepsilon(x) = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left(\frac{x}{\varepsilon}\right)\right)$$

δ_ε has small compact support $[-\varepsilon, \varepsilon]$

Acts only localy, arround the zero - level curve of φ

Only local minima are computed

$\delta_\varepsilon > 0$ everywhere

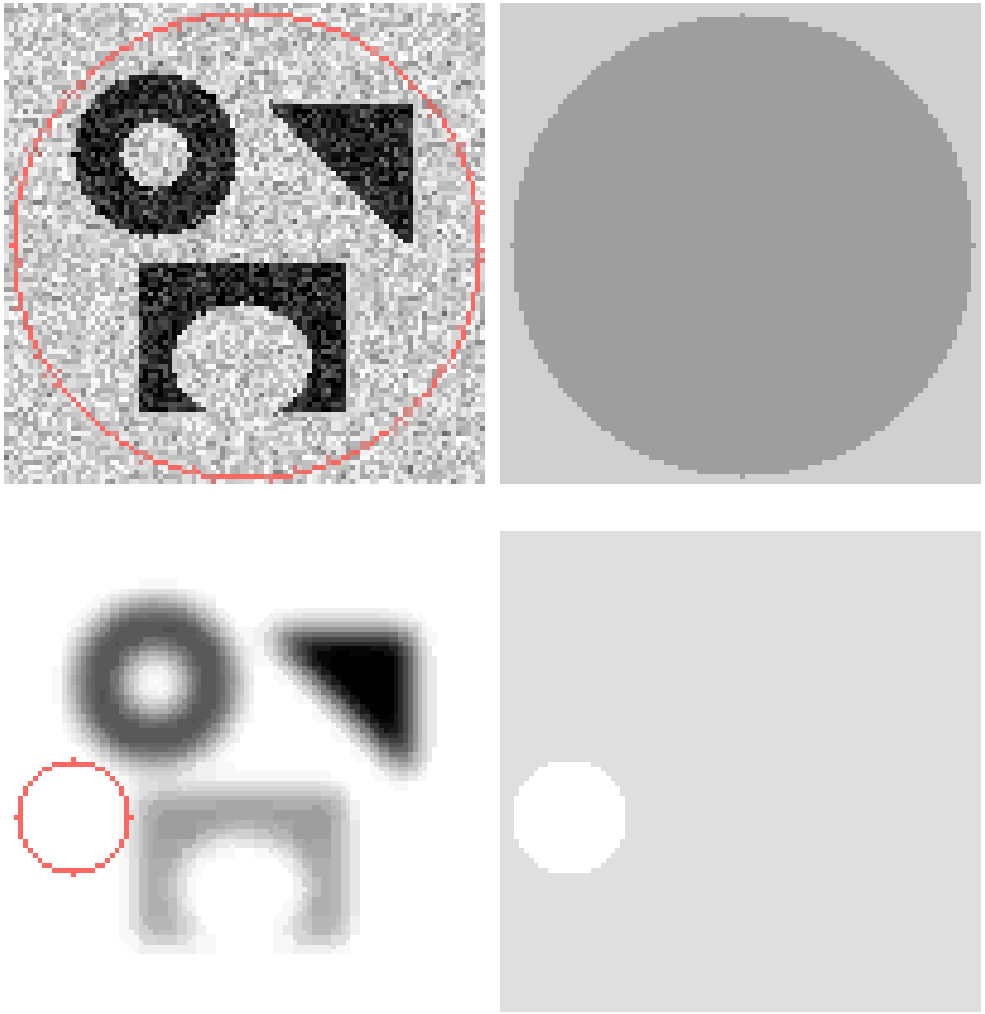
Acts on all level curves of φ

Has the tendency to compute global minima

Detects interior contours automaticaly

Experimental Results

Evolution of C Averages (c_1, c_2)



Advantages

Automatically detects interior contours!

Works very well for concave objects

Robust w.r.t. noise

Detects blurred contours

The initial curve can be placed anywhere!

Allows for automatic change of topolgy

Two linear schemes (fixed point)

Evolutionary iterative scheme

$$\frac{\partial \varphi}{\partial t} = \delta_\varepsilon(\varphi)\left[\mu\nabla\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) - \lambda(u_0 - c_1)^2 + \lambda(u_0 - c_2)^2\right]$$

$$\frac{\varphi^{n+1} - \varphi^n}{\Delta t} = \delta_\varepsilon(\varphi^n)\left[\mu\nabla\left(\frac{\nabla\varphi^{n+1}}{|\nabla\varphi^n|}\right) - \lambda(u_0 - c_1(\varphi^n))^2 + \lambda(u_0 - c_2(\varphi^n))^2\right]$$

Stationary iterative scheme

$$\delta_\varepsilon(\varphi)\left[\mu\nabla\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) - \lambda(u_0 - c_1)^2 + \lambda(u_0 - c_2)^2\right] = 0$$

Our approximation $\delta_\varepsilon(\varphi) > 0$ (strictly positive everywhere)

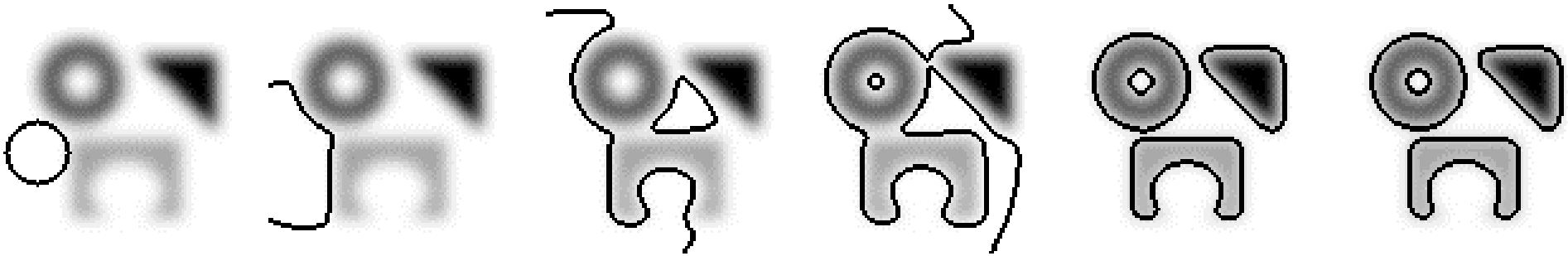
$$\Rightarrow \mu\nabla\left(\frac{\nabla\varphi^{n+1}}{|\nabla\varphi^n|}\right) - \lambda(u_0 - c_1(\varphi^n))^2 + \lambda(u_0 - c_2(\varphi^n))^2 = 0$$

Comparison of the evolutionary/stationary schemes

$$\mu = 0.01 \cdot 255^2, \ \lambda = 1, \ \Delta x = 1$$

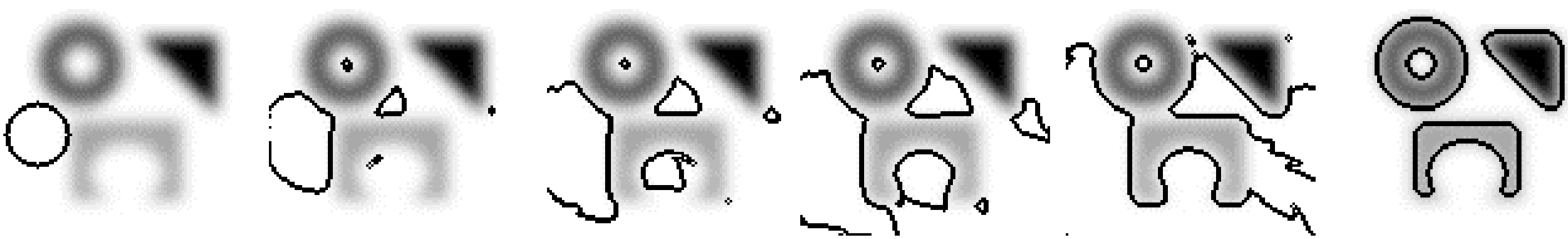
Evolutionary Scheme (CPU time = 59.13 sec) $\Delta t = 0.1, \ \varepsilon = 1$

0 It 50 It 500 It 1000 It 2000 It 4000 It

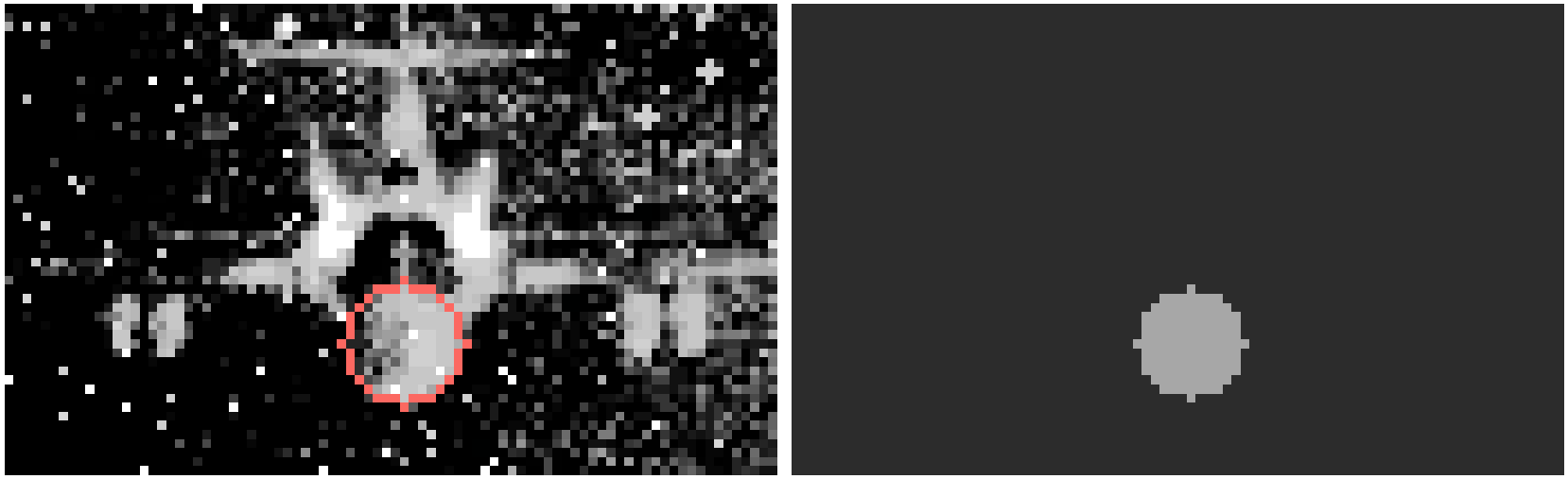


Stationary Scheme (CPU time = 0.63 sec)

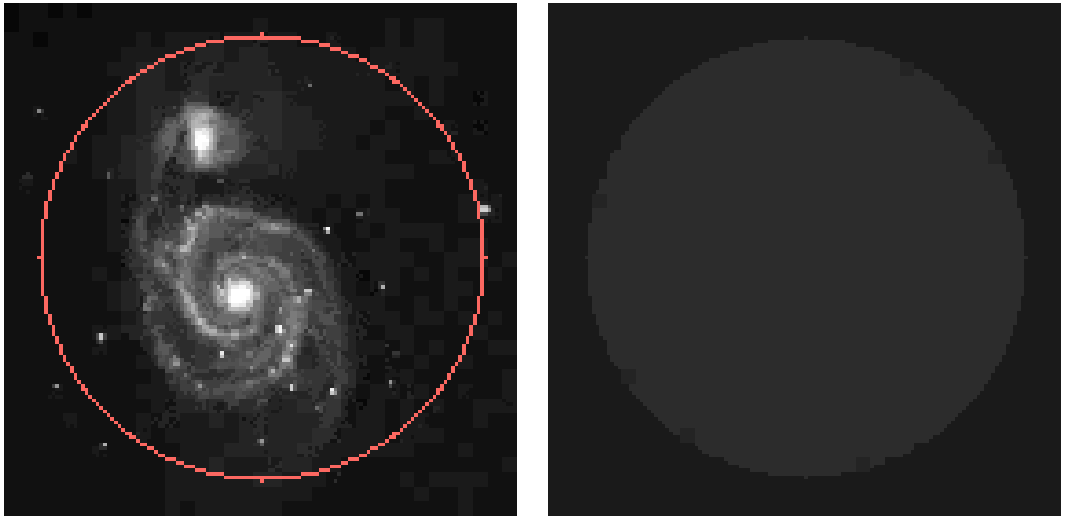
0 It 10 It 20 It 30 It 40 It 50 It



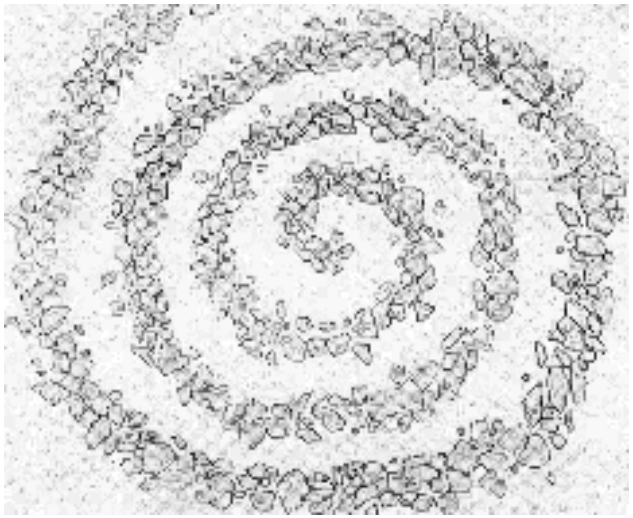
A plane in a noisy environment



A galaxy

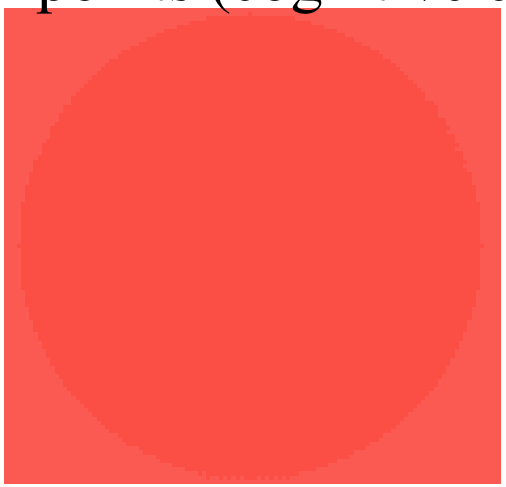


A spiral from an art picture

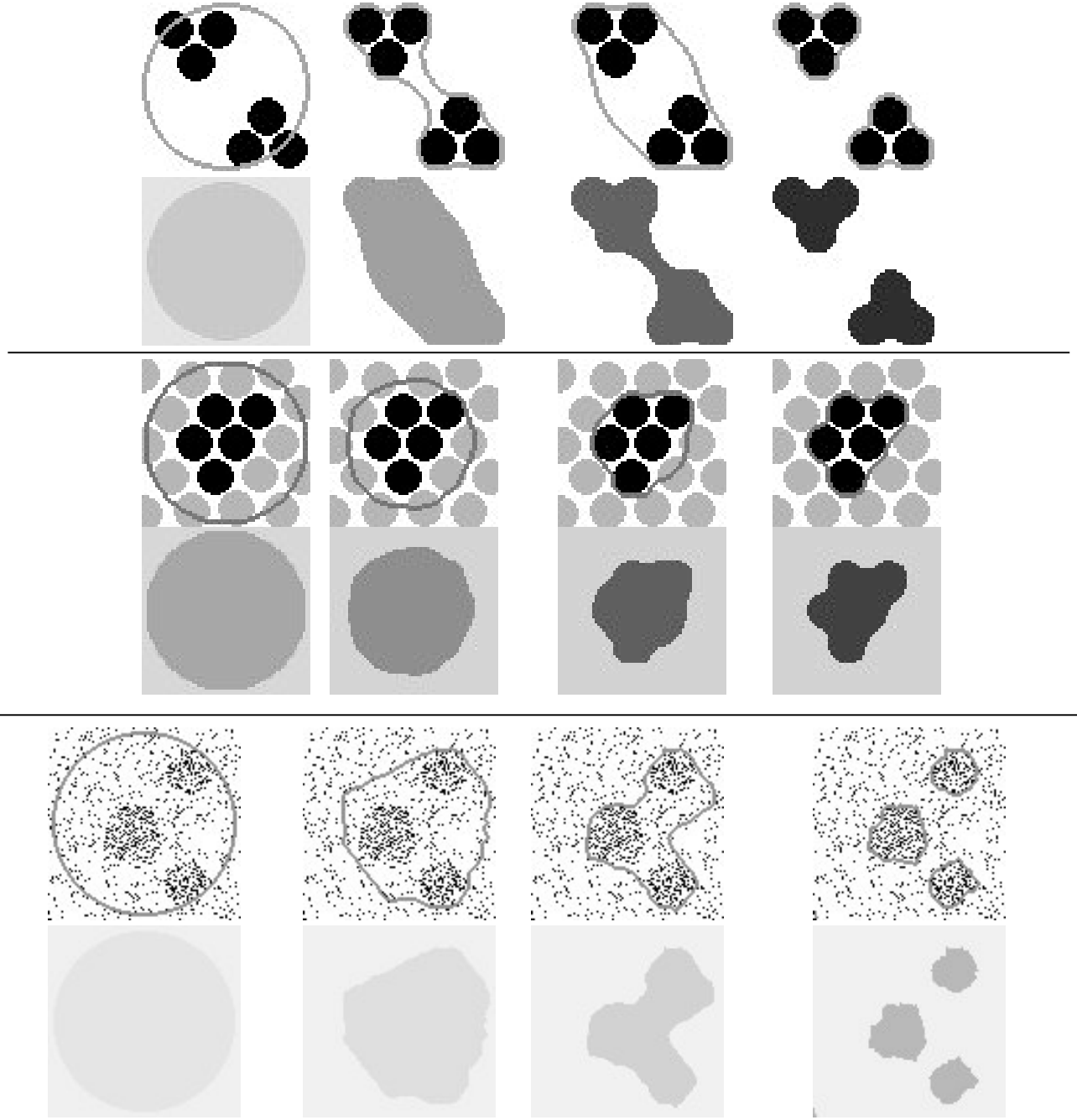


Europe nightlights

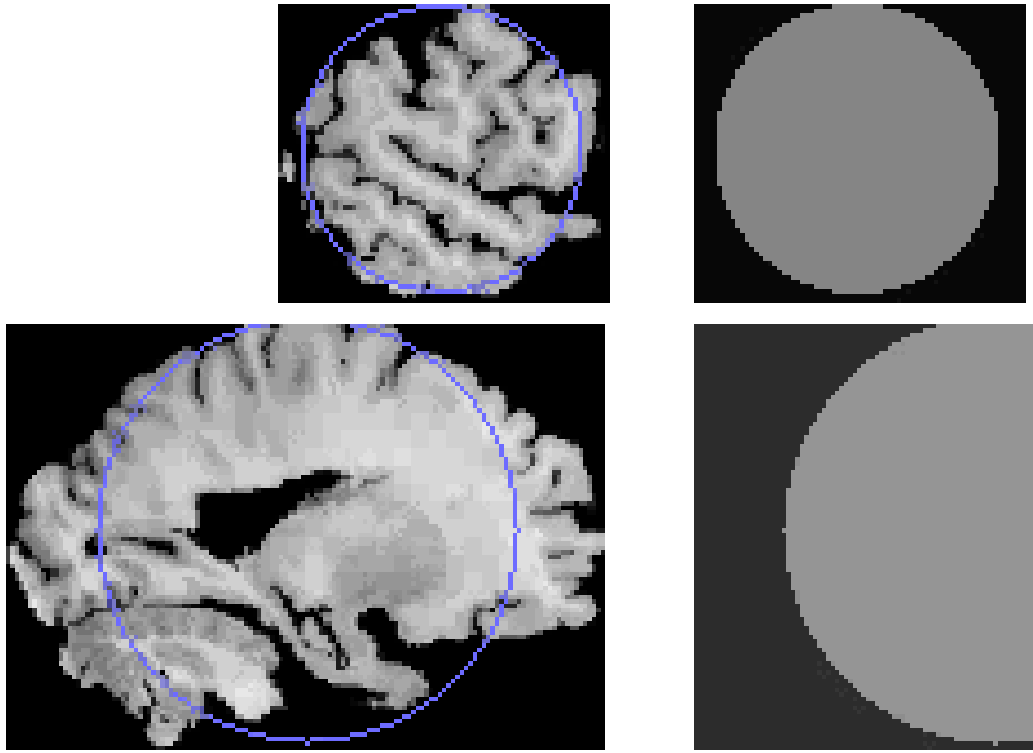
The model detects clusters of points (cognitive contours)



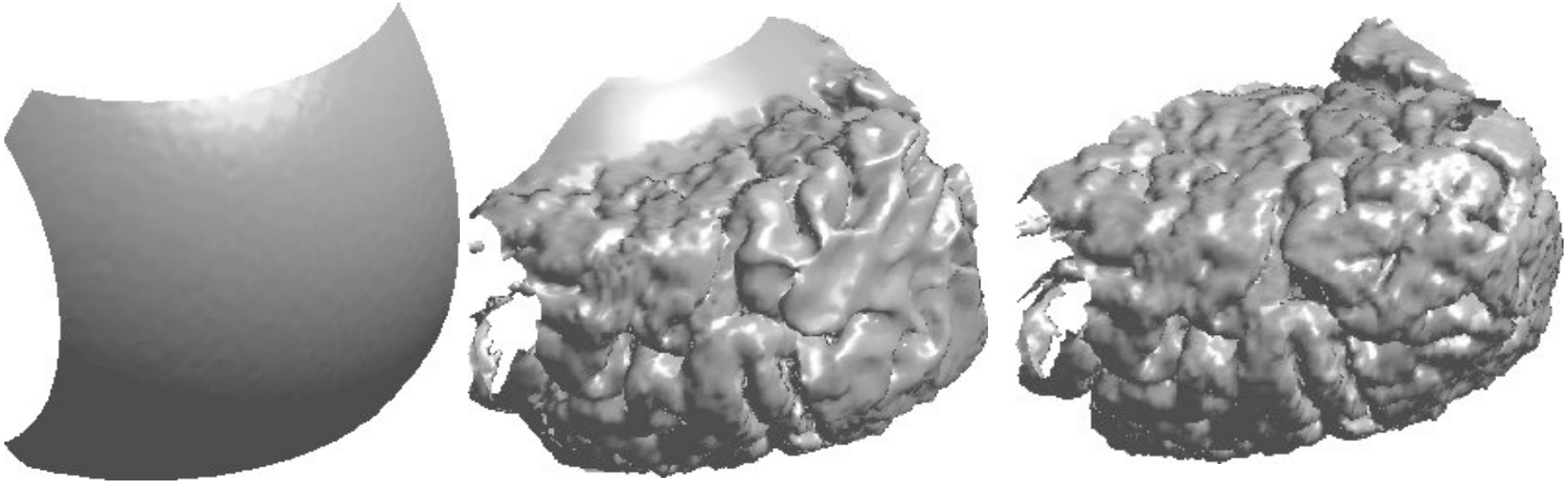
Contours without gradient (cognitive contours)



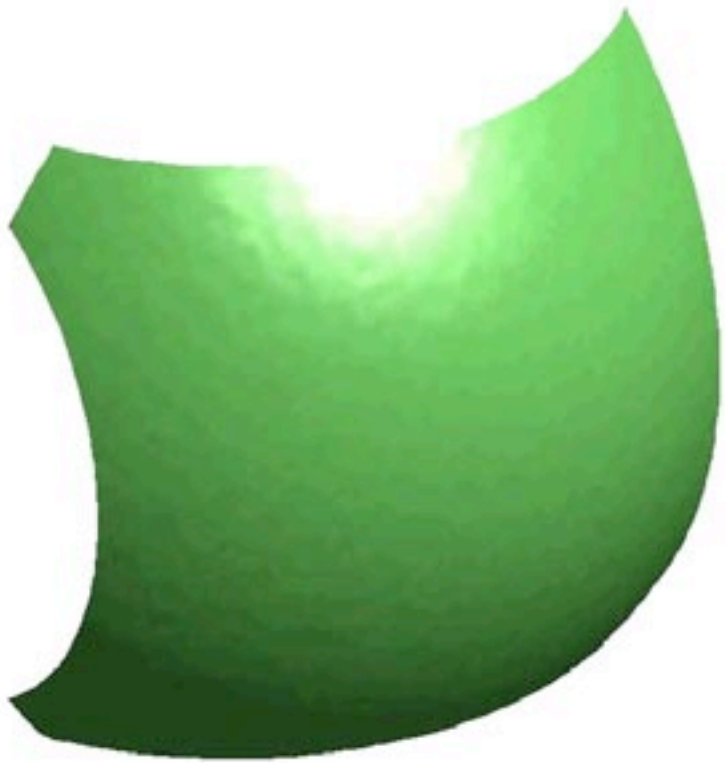
From Active Contours (2D) to Active Surfaces (3D)



MRI DATA FROM LONI-UCLA



Active surface for 3D shape extraction in MRI data



Needs to be improved for medical applications: topology constraint

PLAN

I. Active contours without edges

II. Generalization to the Mumford-Shah segmentation model

- The piecewise-constant case
- The piecewise-smooth case

III. Anisotropic energies:

Total Variation-based active contours using level sets

Anisotropic Mumford-Shah like models

Generalization to Multiple (>2) Segments

The Mumford-Shah piecewise-constant segmentation model

$$F^{MS}(u,C) = \sum_i |u_0 - c_i|^2 \, dxdy + \nu |C|$$

Need > 1 level set function to represent > 2 segments, triple junctions,

Previous Methods for Multiphase Level Set Representation:

(Zhao-Chan-Merriman-Osher, Smith-Chopp):

New Multiphase Level Set Representation:

Using n level set functions, we can represent 2^n phases or segments

Reduced computational cost

No vacuum and no overlap between the phases

Allows for triple junctions and other complex topologies

New multiphase level set representations and partitions

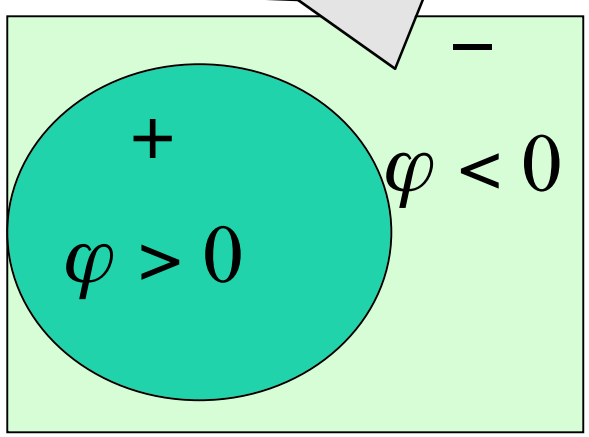
allows for triple junctions, with *no vacuum* and *no overlap* of phases

$$\Phi = (\varphi_1, \ldots, \varphi_n) \quad \Rightarrow \quad 2^n \text{ phases}$$

Curves :

$\{\varphi = 0\}$

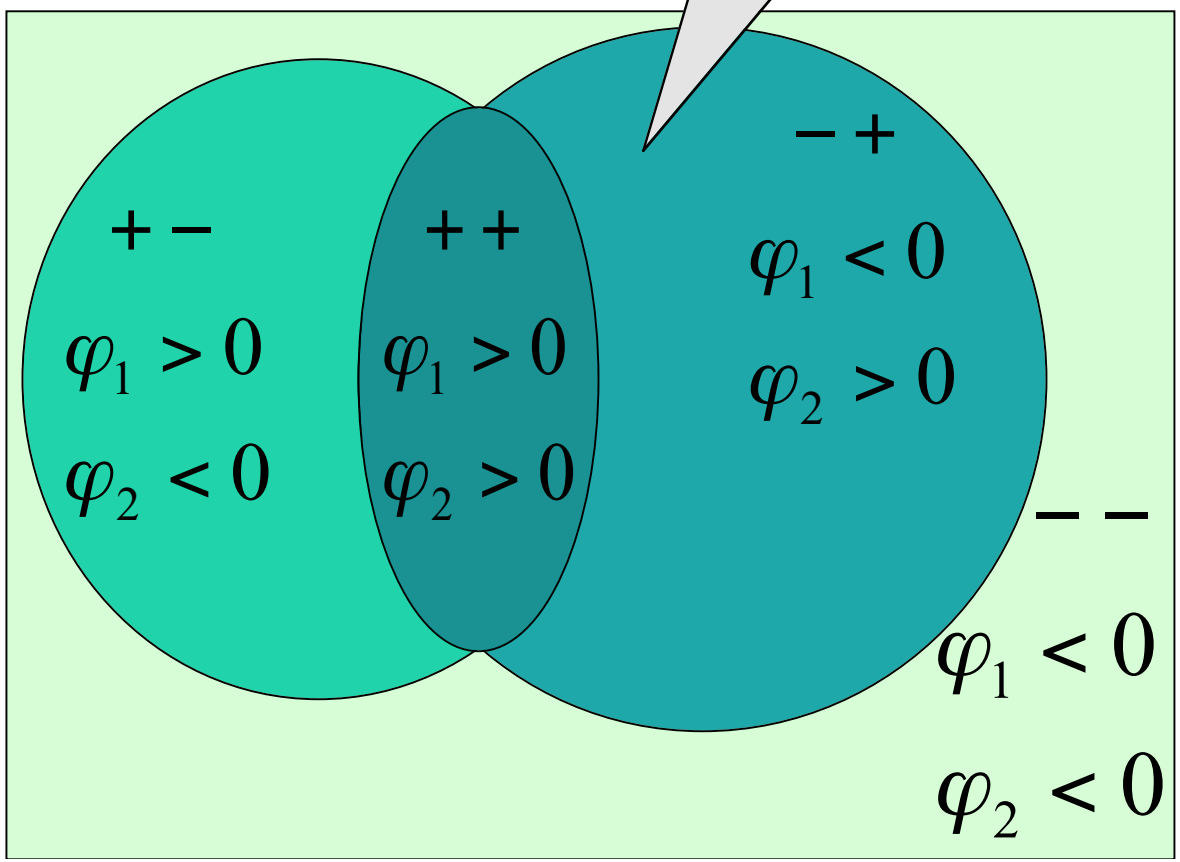
2-phase segmentation
1 level set function



Curves :

$\{\varphi_1 = 0\} \cup \{\varphi_2 = 0\}$

4-phase segmentation
2 level set functions



Example: two level set functions and four phases in the piecewise-constant case

2 level set functions (φ_1, φ_2) $\Leftrightarrow$ 4 phases or segments :

$\varphi_1 > 0, \varphi_2 > 0, \quad \varphi_1 > 0, \varphi_2 < 0, \quad \varphi_1 < 0, \varphi_2 > 0, \quad \varphi_1 < 0, \varphi_2 < 0$

$\Phi = (\varphi_1, \varphi_2)$ The level set functions

$c = (c_{11}, c_{10}, c_{01}, c_{00})$ Constant vector

$$u = c_{11} H(\varphi_1) H(\varphi_2) + c_{10} H(\varphi_1)(1 - H(\varphi_2))$$
$$+ c_{01}(1 - H(\varphi_1)) H(\varphi_2) + c_{00}(1 - H(\varphi_1))(1 - H(\varphi_2))$$

Energy

$$\underset{(c,\Phi)}{Inf} \; F(c,\Phi) = \int_\Omega |u_0 - c_{11}|^2 \, H(\varphi_1) H(\varphi_2) dxdy + \int_\Omega |u_0 - c_{10}|^2 \, H(\varphi_1)(1 - H(\varphi_2)) dxdy$$
$$+ \int_\Omega |u_0 - c_{01}|^2 \, (1 - H(\varphi_1)) H(\varphi_2) dxdy + \int_\Omega |u_0 - c_{00}|^2 \, (1 - H(\varphi_1))(1 - H(\varphi_2)) dxdy$$
$$+ \nu \int_\Omega |\nabla H(\varphi_1)| + \nu \int_\Omega |\nabla H(\varphi_2)|$$

The Euler-Lagrange equations

The constants

$$c_{11}(t) = mean(u_0) \text{ in } \{\varphi_1 > 0, \varphi_2 > 0\}$$
$$c_{10}(t) = mean(u_0) \text{ in } \{\varphi_1 > 0, \varphi_2 < 0\}$$
$$c_{01}(t) = mean(u_0) \text{ in } \{\varphi_1 < 0, \varphi_2 > 0\}$$
$$c_{00}(t) = mean(u_0) \text{ in } \{\varphi_1 < 0, \varphi_2 < 0\}$$

The PDE's in $\Phi = (\varphi_1, \varphi_2)$

$$\frac{\partial \varphi_1}{\partial t} = \delta_\varepsilon(\varphi_1)$$
$$\left\{ \nu div\left(\frac{\nabla \varphi_1}{|\nabla \varphi_1|}\right) - \left[\left(|u_0 - c_{11}|^2 - |u_0 - c_{01}|^2\right) H(\varphi_2) + \left(|u_0 - c_{10}|^2 - |u_0 - c_{00}|^2\right)(1 - H(\varphi_2))\right] \right\}$$
$$\frac{\partial \varphi_2}{\partial t} = \delta_\varepsilon(\varphi_2)$$
$$\left\{ \nu div\left(\frac{\nabla \varphi_2}{|\nabla \varphi_2|}\right) - \left[\left(|u_0 - c_{11}|^2 - |u_0 - c_{01}|^2\right) H(\varphi_1) + \left(|u_0 - c_{10}|^2 - |u_0 - c_{00}|^2\right)(1 - H(\varphi_1))\right] \right\}$$

Similarly for $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$

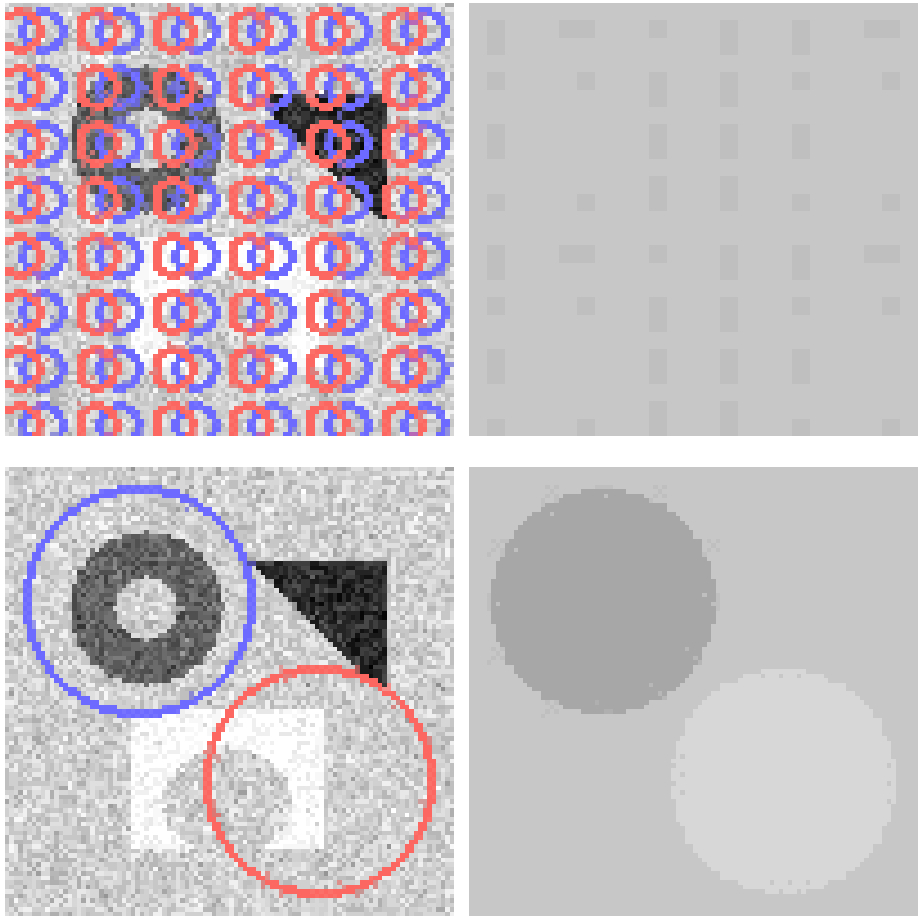
Results with 2 level set functions (4 phases)

Two different initializations

Interior contours are automatically detected

Model robust to noise

Faster



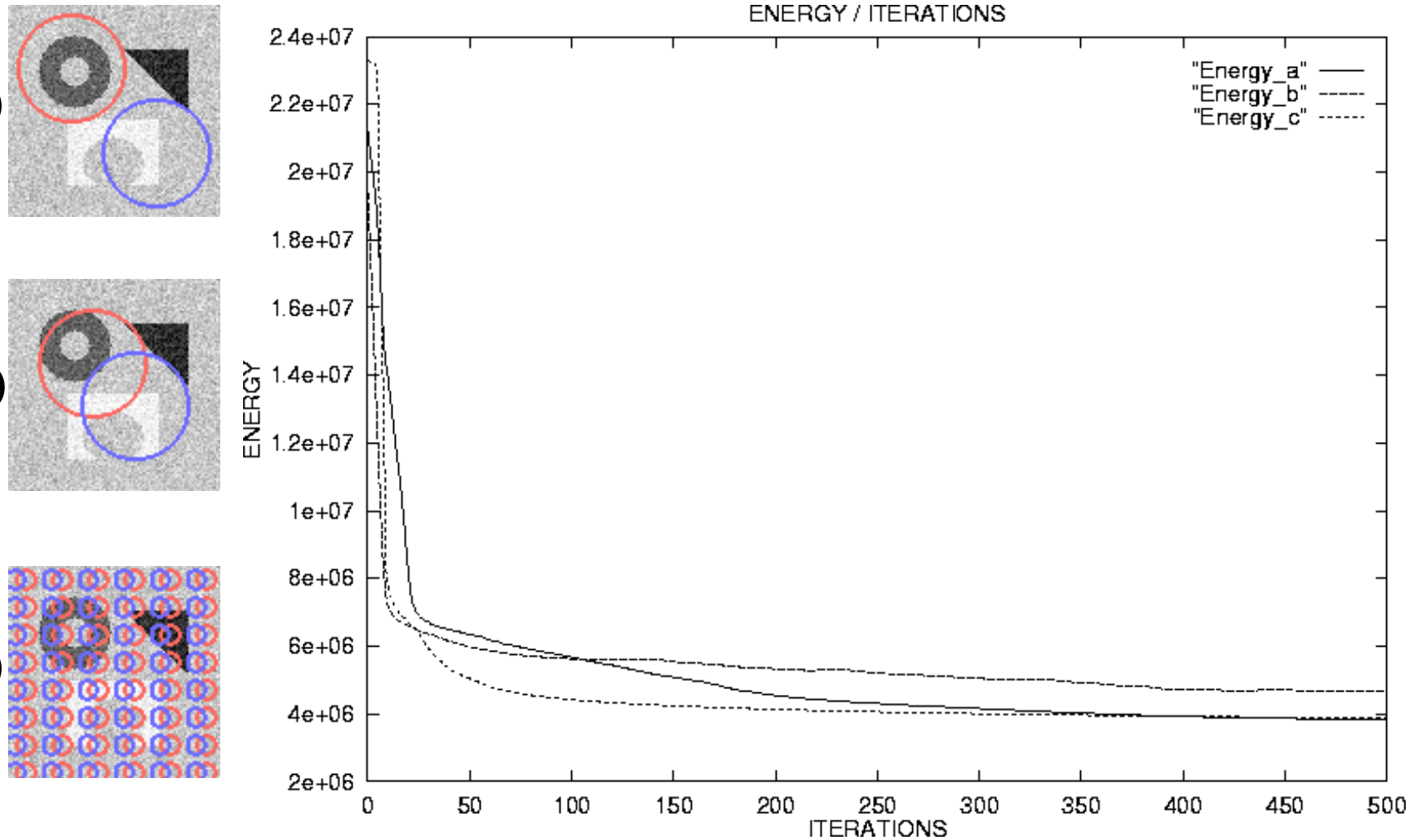
Energy Decrease With Iterations for 3 initial conditions

(here, only with (a) and (c) it converge to the global minimum)

(a)

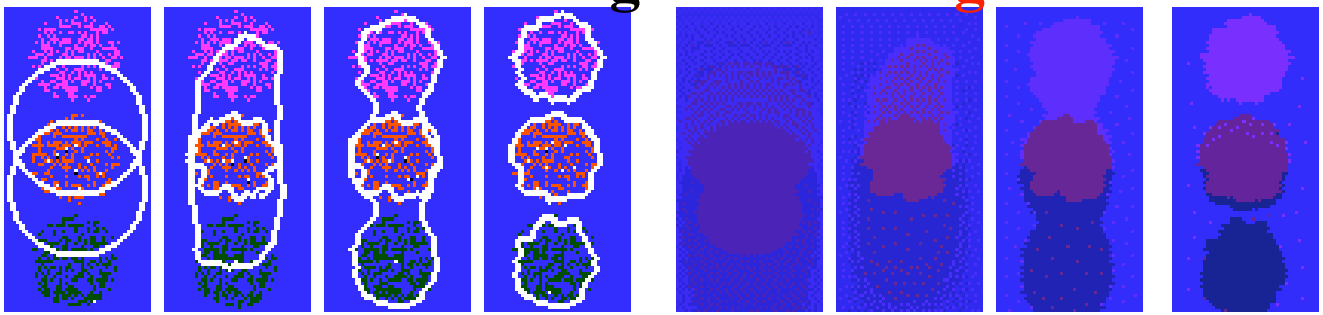
(b)

(c)

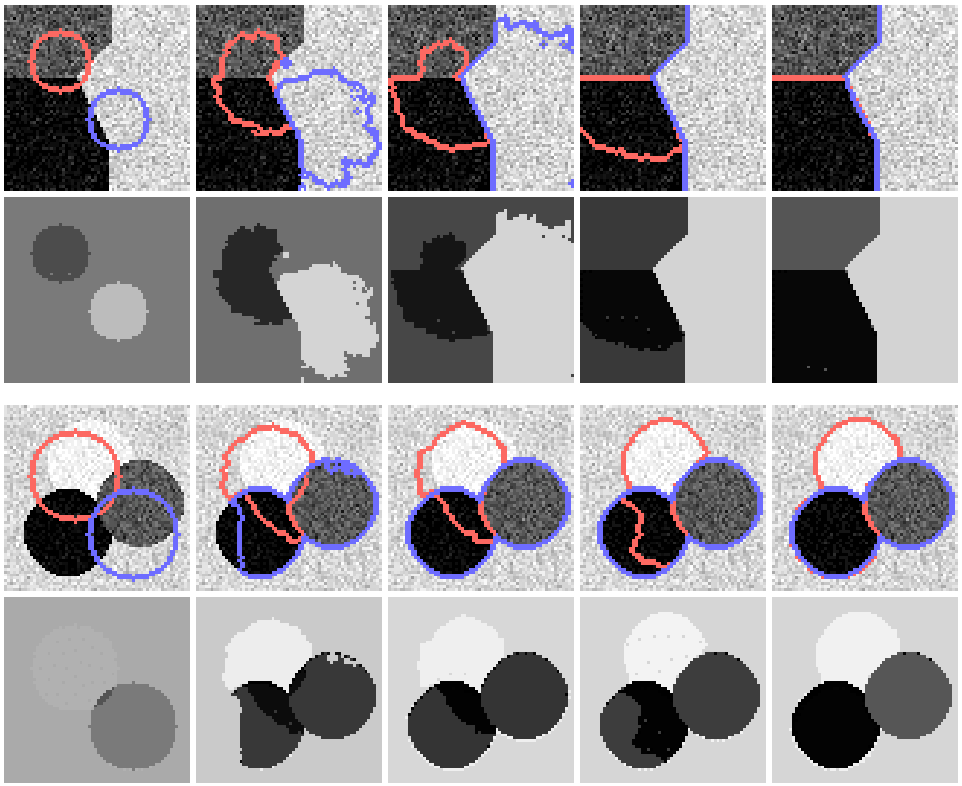


Non-convex problem: different (I.C.) may converge to local/global min.

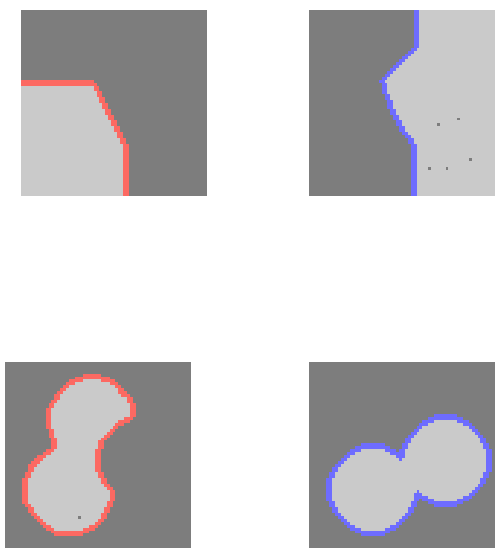
Detection of contours without gradient - cognitive contours



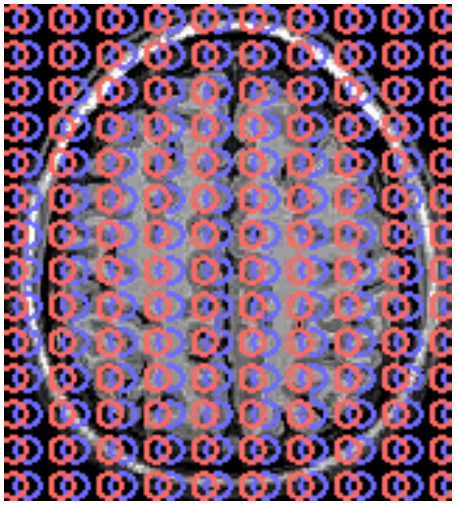
Triple junctions can be detected and represented using only two level set functions, without vacuum and without overlap.

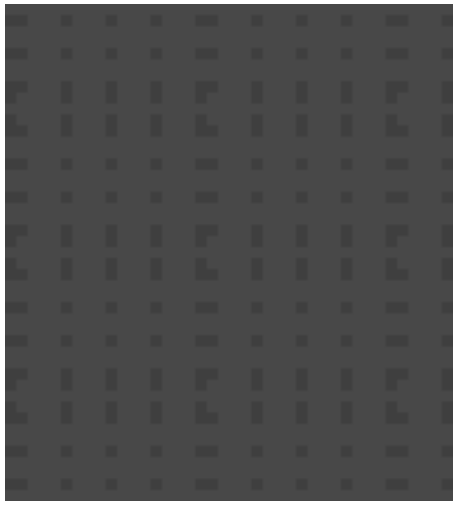


$\{\varphi_1 = 0\}$ $\{\varphi_2 = 0\}$



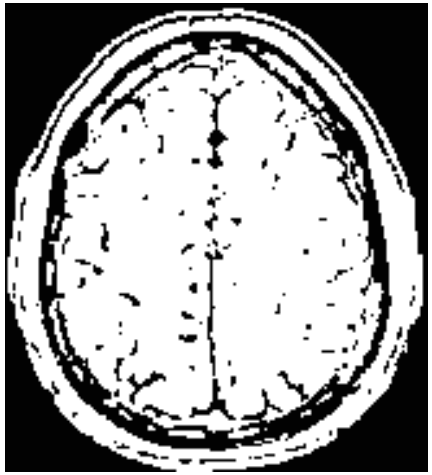
A brain image

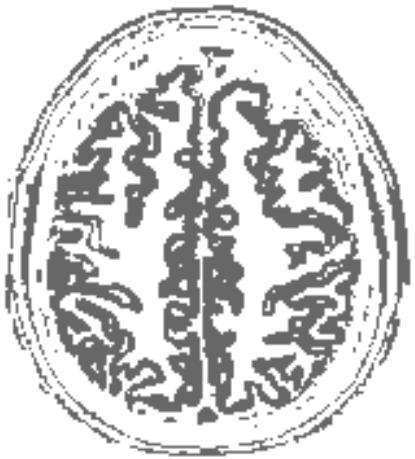




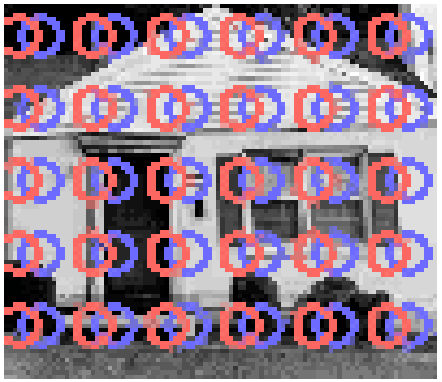
Phase 11 Phase 10 Phase 01 Phase 00

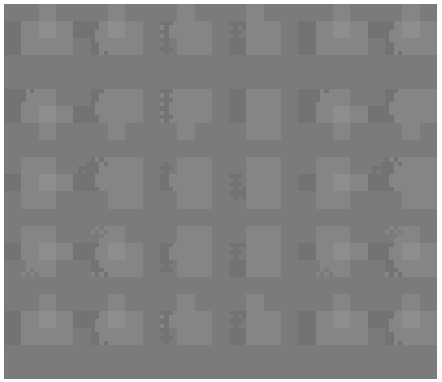
mean(11)=45 mean(10)=159 mean(01)=9 mean(00)=103





A real outdoor picture

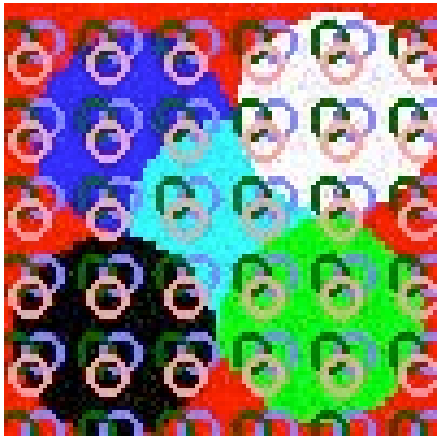




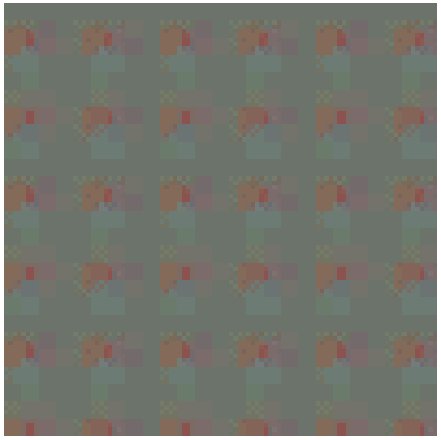
Phase 11	Phase 10	Phase 01	Phase 00
mean(11)=159	mean(10)=205	mean(01)=23	mean(00)=97

Three level set functions representing up to eight phases

Six phases are detected, together with the triple junctions

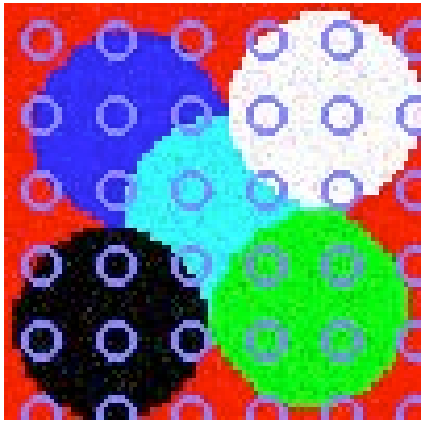


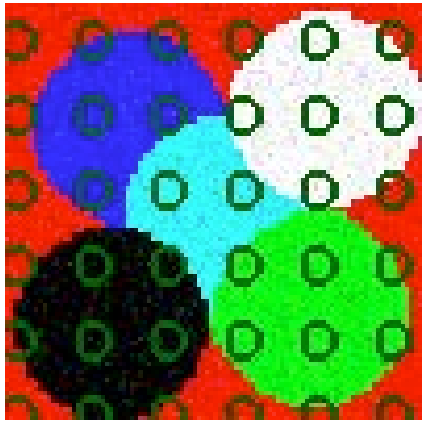
Evolution of the 3 level sets

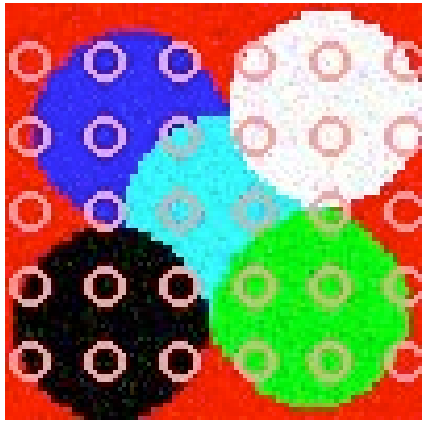


Segmented image

T







Evolution of the 3 individual level sets.

Generalization to Piecewise-Smooth Mumford-Shah segmentation model

The Mumford-Shah segmentation model 89

$$F^{MS}(u,C) = \mu^2 \int_{\Omega} |u - u_0|^2 \, dxdy + \int_{\Omega - C} |\nabla u|^2 \, dxdy + \nu \, |C|$$

A level set algorithm for the M-S model

Assume: C= boundary of an open domain; $C = \{(x,y) \in \Omega \,|\, \varphi(x,y) = 0\}$

Related work by L. Cohen et al., Yezzi et al.

$$u(x,y) = \begin{cases} u^+(x,y), \text{ if } \varphi \geq 0 \\ u^-(x,y), \text{ if } \varphi < 0 \end{cases}$$

$$u(x,y) = u^+(x,y)H(\varphi) + u^-(x,y)(1 - H(\varphi))$$

$$F(u^+,u^-,\varphi) = \mu^2 \int_{\Omega} |u^+ - u_0|^2 \, H(\varphi)dxdy + \mu^2 \int_{\Omega} |u^- - u_0|^2 \, (1 - H(\varphi))dxdy$$

$$+ \int_{\Omega} |\nabla u^+|^2 \, H(\varphi)dxdy + \int_{\Omega} |\nabla u^-|^2 \, (1 - H(\varphi))dxdy + \nu \int_{\Omega} |\nabla H(\varphi)|$$

The Euler-Lagrange equations

$$\mu^2(u^+ - u_0) = \Delta u^+ \text{ on } \varphi > 0, \frac{\partial u^+}{\partial \vec{n}} = 0 \text{ on } \{\varphi = 0\}$$

$$\mu^2(u^- - u_0) = \Delta u^- \text{ on } \varphi < 0, \frac{\partial u^-}{\partial \vec{n}} = 0 \text{ on } \{\varphi = 0\}$$

$$\varphi_t = \delta_\varepsilon(\varphi)\left[\nu \cdot div\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) - \mu^2|u^+ - u_0|^2 + \mu^2|u^- - u_0|^2 - |\nabla u^+|^2 + |\nabla u^-|^2\right]$$

Common framework for active contours, denoising, segmentation and edge detection.

To handle triple junctions and more general cases

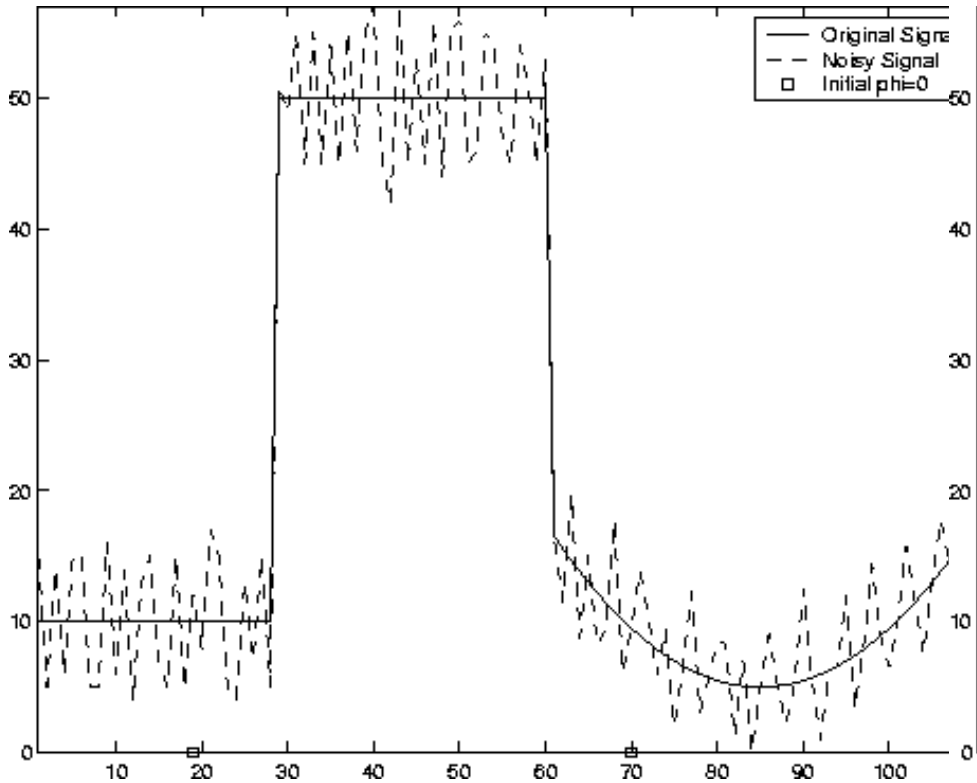
$$u = u^{++}H(\varphi_1)H(\varphi_2) + u^{+-}H(\varphi_1)(1 - H(\varphi_2)) +$$
$$u^{-+}(1 - H(\varphi_1))H(\varphi_2) + u^{--}(1 - H(\varphi_1))(1 - H(\varphi_2))$$

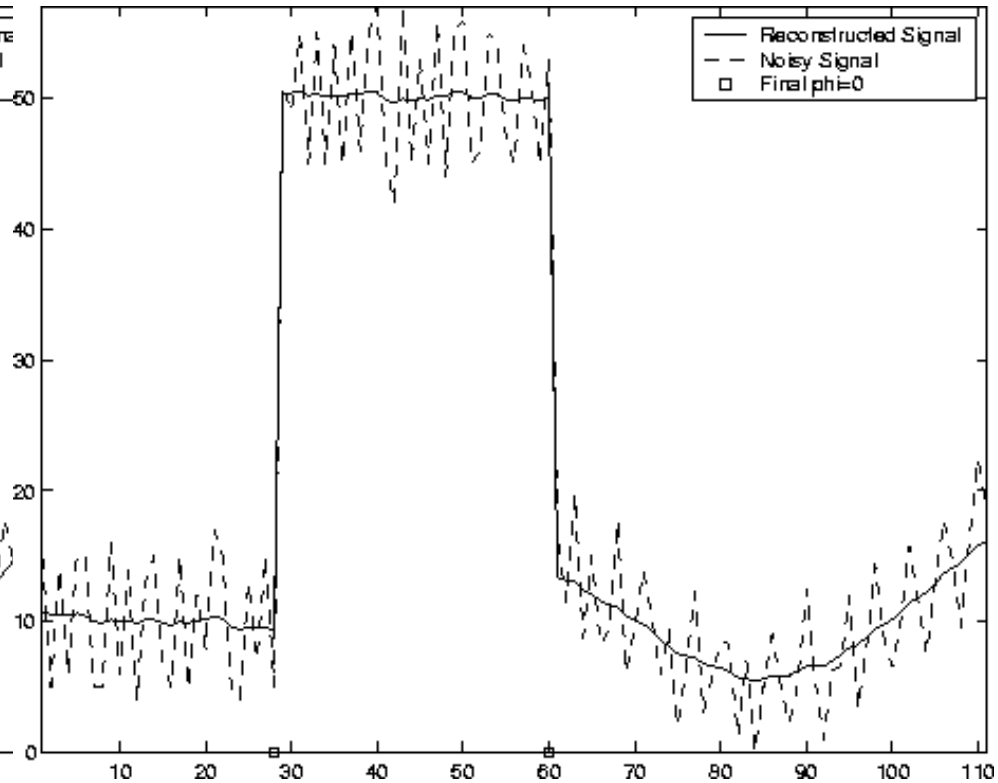
Remark: Based on the Four Color Thm., it should suffice

The energy and the Euler-Lagrange equations are similar

Signal denoising - segmentation - jump detection

(numerical results in 1D; note jumps well located and preserved)

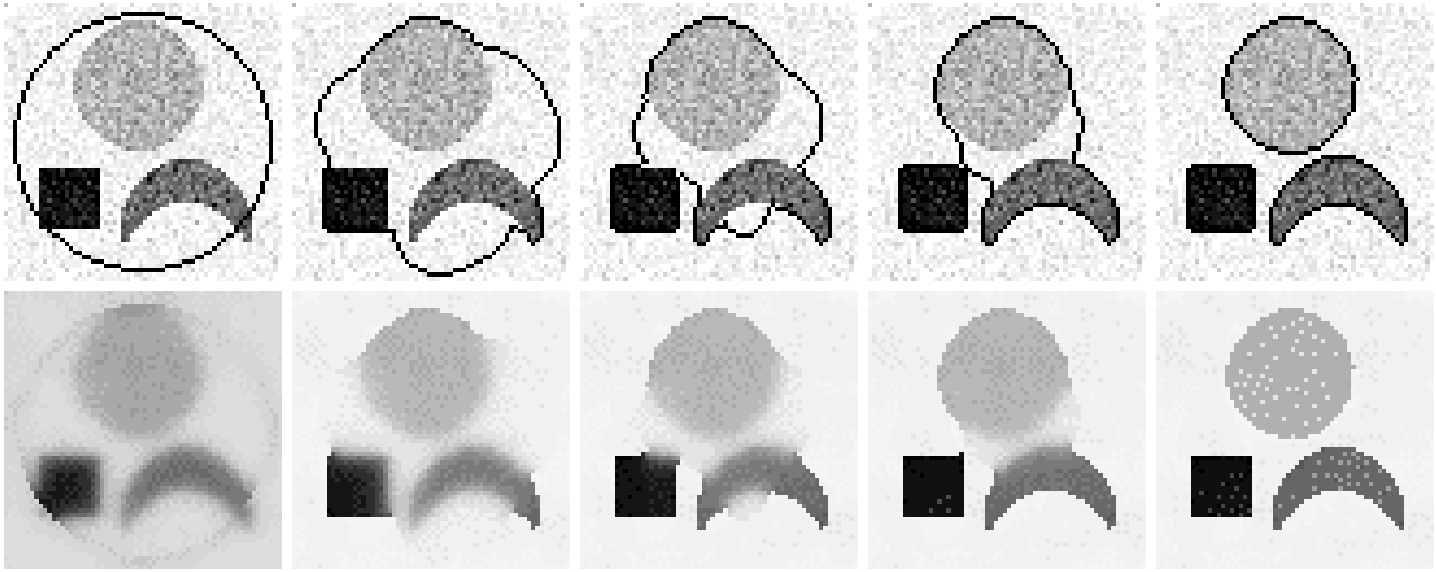


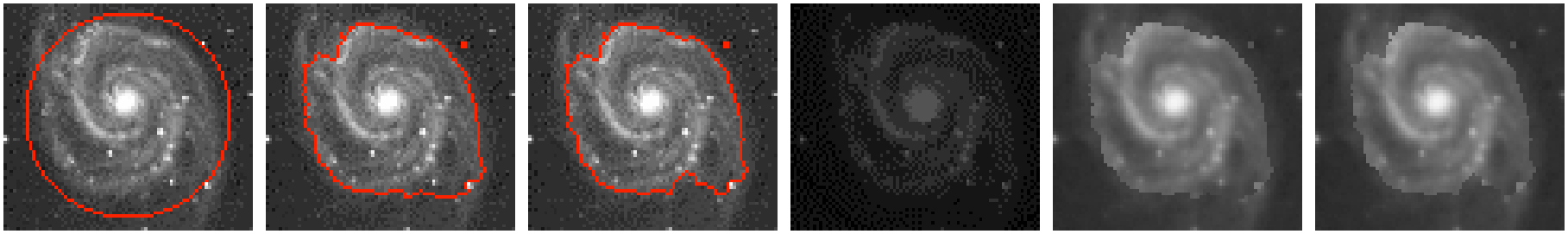


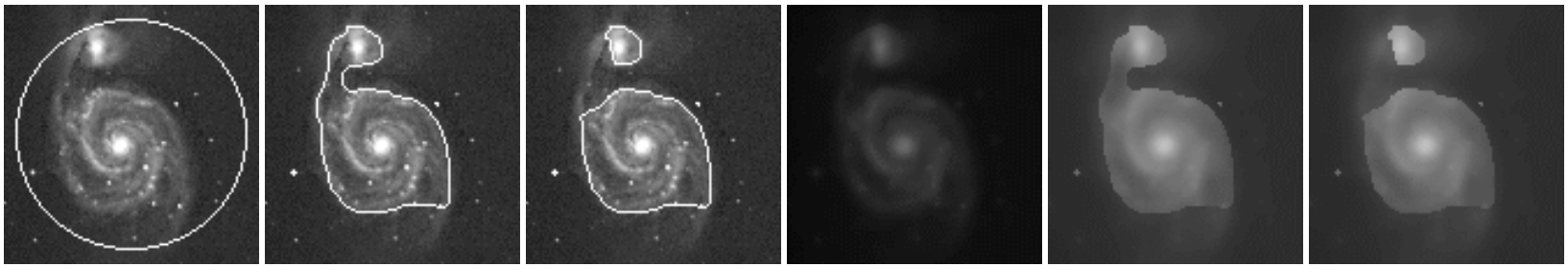
Remark: Related works by Anne Gelb and Eitan Tadmor using spectral methods

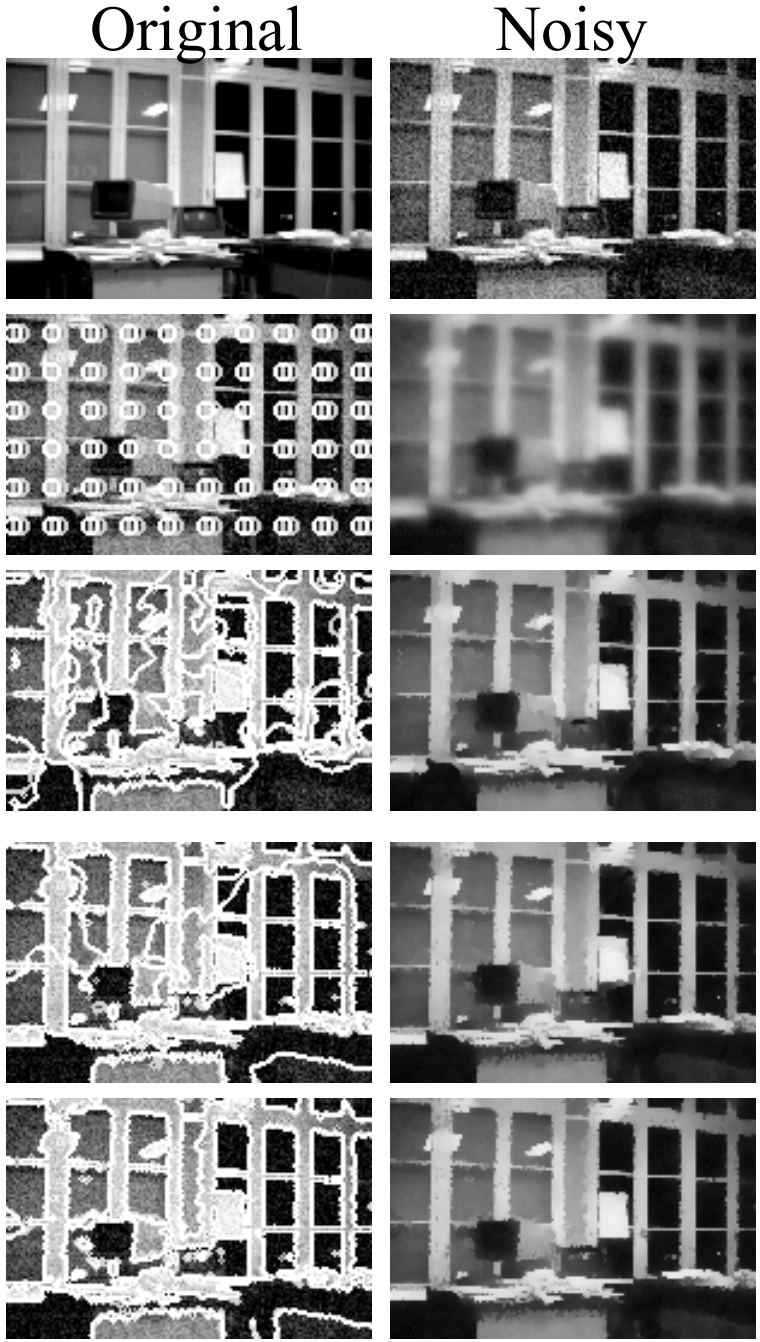
Piecewise-smooth segmentation

Active Contours+Denoising+Segmentation









$$\left\{\begin{aligned} u(x) &= u^{++}(x)H(\varphi_1)H(\varphi_2) \\ &+ u^{+-}(x)H(\varphi_1)(1-H(\varphi_2)) \\ &+ u^{-+}(x)(1-H(\varphi_1))H(\varphi_2) \\ &+ u^{--}(x)(1-H(\varphi_1))(1-H(\varphi_2)) \end{aligned}\right\}$$

PLAN

I. Active contours without edges

II. Generalization to the Mumford-Shah segmentation model

- The piecewise-constant case
- The piecewise-smooth case

III. Anisotropic energies (brief)

Total Variation-based active contours using level sets

Anisotropic Mumford-Shah like models

Summary:

We have minimized Mumford-Shah restricted to sets:

$$\{u = c^{+}H(\varphi) + c^{-}(1 - H(\varphi)),\ \text{with}\ \ c^{+}, c^{-}\ \ \text{constants}\}$$

$$\{u = u^{+}(x)H(\varphi) + u^{-}(x)(1 - H(\varphi)),\ \text{with}\ \ u^{+}, u^{-}\ \ C^{1}\ \text{functions}\}$$

$$\{u = u^{++}(x)H(\varphi_1)H(\varphi_2) + u^{+-}(x)H(\varphi_1)(1 - H(\varphi_2))$$
$$+ u^{-+}(x)(1 - H(\varphi_1))H(\varphi_2) + u^{--}(x)(1 - H(\varphi_1))(1 - H(\varphi_2))\}$$

→ *Curve evolution, object detection, segmentation models*

We can considers restrictions to other sets:

piecewise-polynomials (example: piecewise-linear)

We can consider other energies anisotropic

Example: Total Variation restricted to $\{u = c^+ H(\varphi) + c^-(1 - H(\varphi))\}$

$$\inf_{c^+, c^-, \varphi} \quad F^{TV}(c^+, c^-, \varphi) = \int_\Omega |u_0(x) - c^+|^2 \, H(\varphi) dx$$

$$+ \int_\Omega |u_0(x) - c^-|^2 \, (1 - H(\varphi)) dx + \mu \, |c^+ - c^-| \int_\Omega |\nabla H(\varphi)| \, dx$$

(jump=new scale feature)

=> **Euler-Lagrange equations** (embedded in a dynamic scheme)

$$\varphi(0, x) = \varphi_0(x)$$

$$c^+(t) = average(u_0 |_{\{\varphi(t,x)>0\}}) - \mu \operatorname{sgn}(c^+ - c^-) \frac{Length\{\varphi = 0\}}{Area\{\varphi > 0\}}$$

$$c^-(t) = average(u_0 |_{\{\varphi(t,x)<0\}}) - \mu \operatorname{sgn}(c^- - c^+) \frac{Length\{\varphi = 0\}}{Area\{\varphi < 0\}}$$

$$\frac{\partial \varphi}{\partial t} = \delta_\varepsilon(\varphi) \left[\mu \, |c^+ - c^-| \, div\left(\frac{\nabla \varphi}{|\nabla \varphi|} \right) - |u_0 - c^+|^2 + |u_0 - c^-|^2 \right]$$

(joint work with S. Osher) **(S. Esedoglu: theoretical results)**

More general Mumford and Shah-like functionals:

$$\inf_{u,C} F^{MS-GEN}(u,C) = \int_{\Omega} |u-u_0|^2\, dx + \mu \int_{\Omega - C} f(\nabla u)dx + \nu \int_C g(|u^+ - u^-|)dH^1$$

where

- $p \geq 1,\ f : R^2 \to [0,\infty)$ is convex and s.t. $\lim_{|z|\to\infty} \frac{f(z)}{|z|} = +\infty$
- $g : [0,\infty) \to [0,\infty)$ is sub-additive and increasing s.t. $\lim_{t\to 0} \frac{g(t)}{t} = \infty$

Examples:

$p = 2,\ f(\nabla u) = |\nabla u|^q$ with $q > 1,\ g(|u^+ - u^-|) = \sqrt{|u^+ - u^-|}$

Anisotropic Mumford -Shah-like functional:

$$\inf_{u,C} F^{MS-ANISOTROPIC}(u,C) = \int_{\Omega} |u-u_0|^2\, dx + \mu \int_{\Omega - C} |\nabla u|^2\, dx + \nu \int_C \sqrt{|u^+ - u^-|}dH^1$$

$\Rightarrow$ Mumford-Shah model can also have a scale function of jump!

(still non-convex problem, but removes limitations on type of edges)

Example: **Anisotropic Mumford -Shah** restricted to

$$\{u = c^+ H(\varphi) + c^-(1 - H(\varphi))\}$$

$$\inf_{c^+, c^-, \varphi} F^{MS-ANISOTROPIC}(c^+, c^-, \varphi) = \int_\Omega |u_0(x) - c^+|^2 H(\varphi) dx$$

$$+ \int_\Omega |u_0(x) - c^-|^2 (1 - H(\varphi)) dx + \mu \sqrt{|c^+ - c^-|} \int_\Omega |\nabla H(\varphi)| dx$$

(jump=scale feature)

=> **Euler-Lagrange equations** (embedded in a dynamic scheme)

$$\varphi(0, x) = \varphi_0(x)$$

$$c^+(t) = average(u_0 |_{\{\varphi(t,x)>0\}}) - \mu \frac{\operatorname{sgn}(c^+ - c^-)}{4\sqrt{|c^+ - c^-|}} \frac{Length\{\varphi = 0\}}{Area\{\varphi > 0\}}$$

$$c^-(t) = average(u_0 |_{\{\varphi(t,x)<0\}}) - \mu \frac{\operatorname{sgn}(c^- - c^+)}{4\sqrt{|c^+ - c^-|}} \frac{Length\{\varphi = 0\}}{Area\{\varphi < 0\}}$$

$$\frac{\partial \varphi}{\partial t} = \delta_\varepsilon(\varphi) \left[\mu \sqrt{|c^+ - c^-|} div\left(\frac{\nabla \varphi}{|\nabla \varphi|} \right) - |u_0 - c^+|^2 + |u_0 - c^-|^2 \right]$$