LosAlamos 2002

The Euler Equations of Computational Anatomy

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Trouvé, Younes Euler-Lagrange Equations of Computational Anatomy 2002 Annual Review BME

What Should Computational Anatomy Be?


This is Elizabeth and Michael’s baby girl,
age 1-2,
with Michael’s eyes,
Elizabeth’s nose,
she is beautiful,
and …….

Grenander’s Metric Pattern Theory

"In a very large part of morphology, our essential task lies in the comparison of related forms rather in the precise definition of each; …. This process of comparison, of recognizing in one form a definite permutation or deformation of another, … is the Method of Coordinates,on which is based the Theory of Transformations".

D'Arcy Thompson, On Growth and Form, 1917.

Mathematical Model of Anatomy:

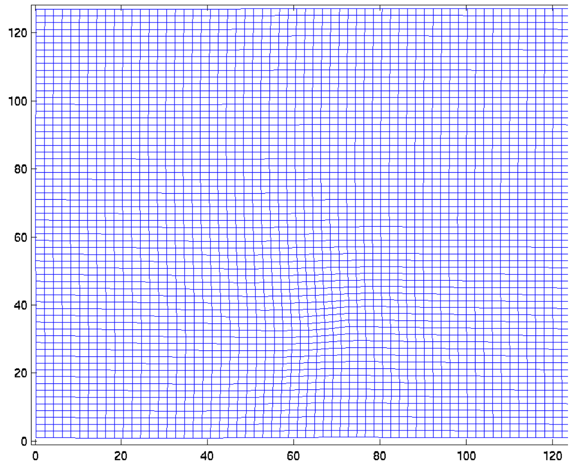
Anatomy is an Orbit (deformable template) Under Groups of Diffeomorphisms $G = \{\phi : X \leftrightarrow X\}$

Images are functions $I(x), x \in X$

Anatomy is all images generated from an exemplar template

$$ANATOMY = \underbrace{G \cdot I_{temp}}_{group-action} = \left\{ \underbrace{I_{temp} \circ \phi}_{function-composition}, \phi \in G \right\}$$

Grenander, Miller, Quarterly Applied Math. 1998



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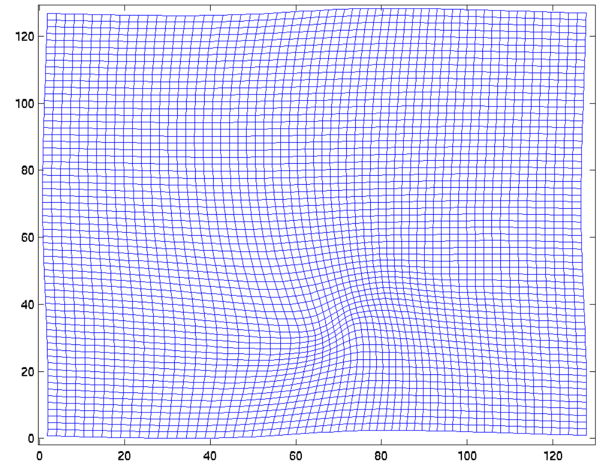

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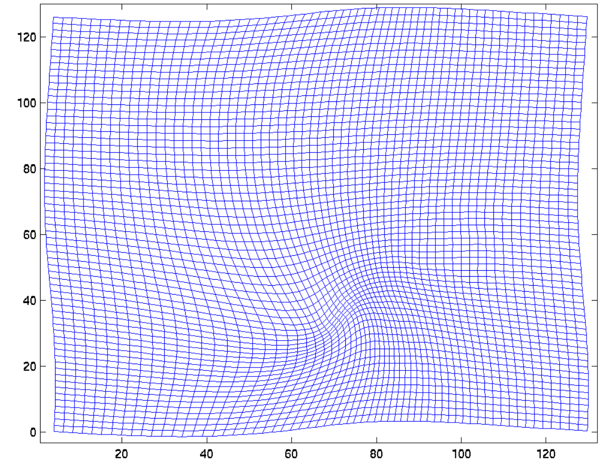

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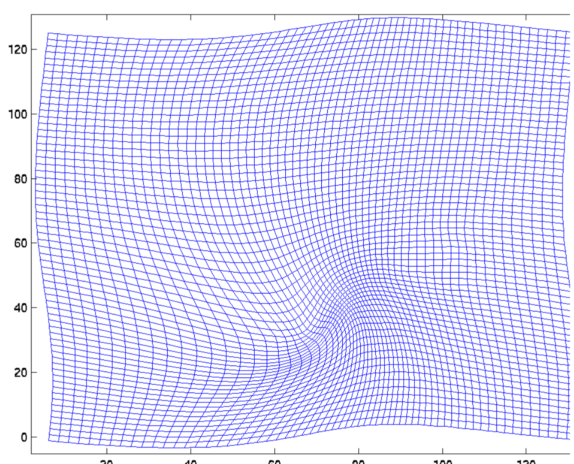
$$\phi : X \leftrightarrow X$$



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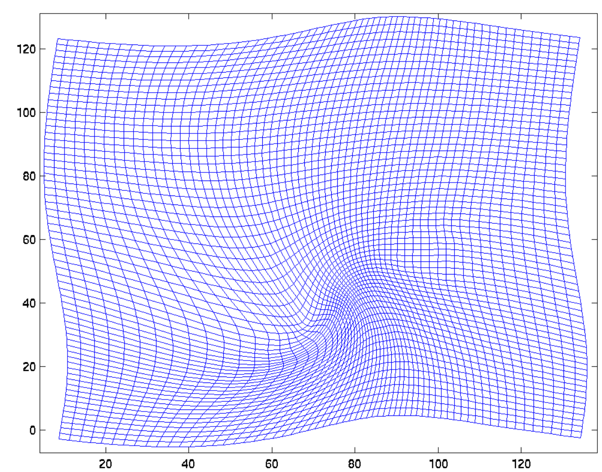

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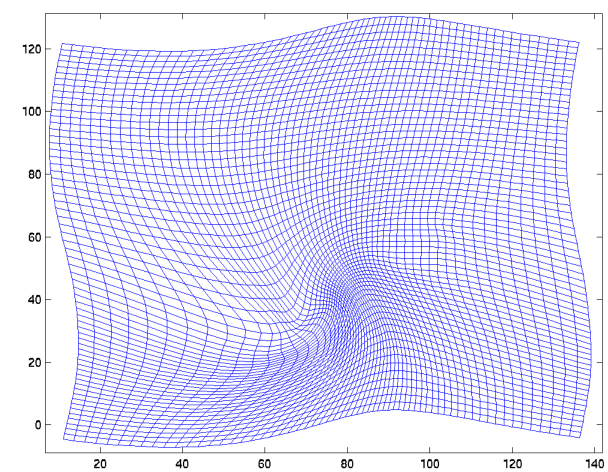

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The Orbit of Mitochondria

$$G \cdot I_{temp} = \left\{ I_{temp} \circ \phi, \phi \in G \right\}$$

jump

Computational Anatomy: An Emerging Discipline

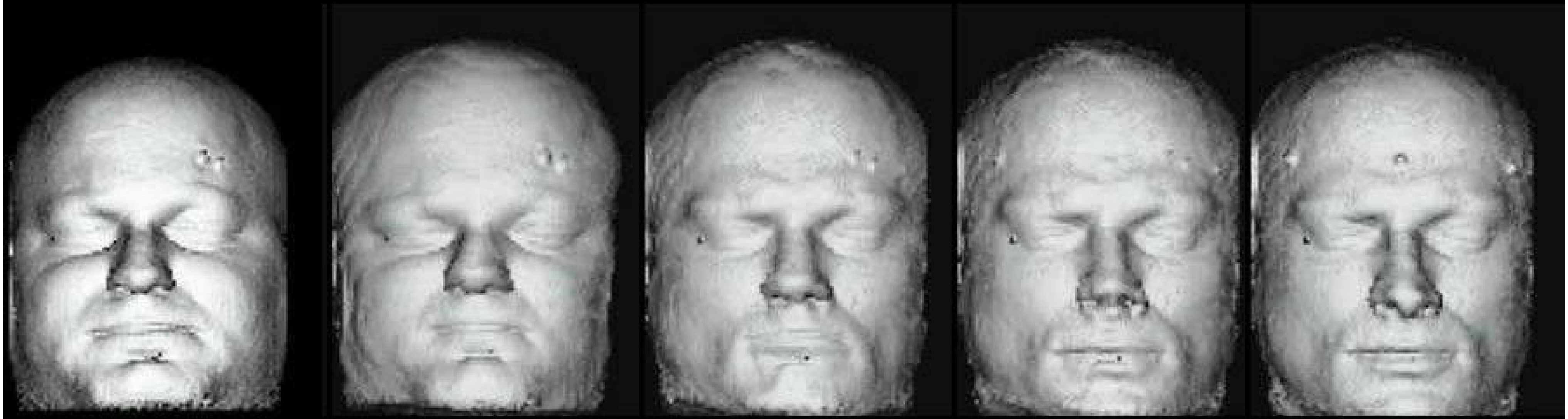
Constructing the Manifolds of Anatomical Structures
(landmarks, curves, surfaces, subvolumes)

Comparing the Manifolds of Anatomical Structures
(metric spaces for anatomies – Gren.,Trouve,Younes)

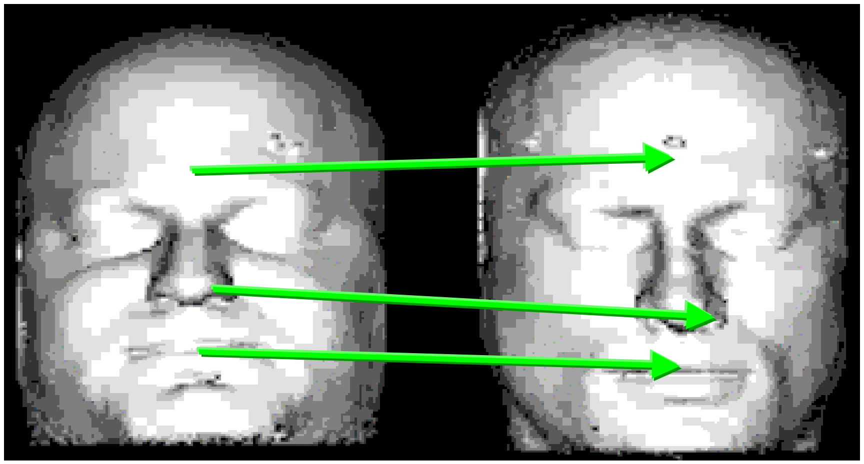
Construct probabilities representing anatomical variation
(empirical distribution construction
disease or large deviation testing)

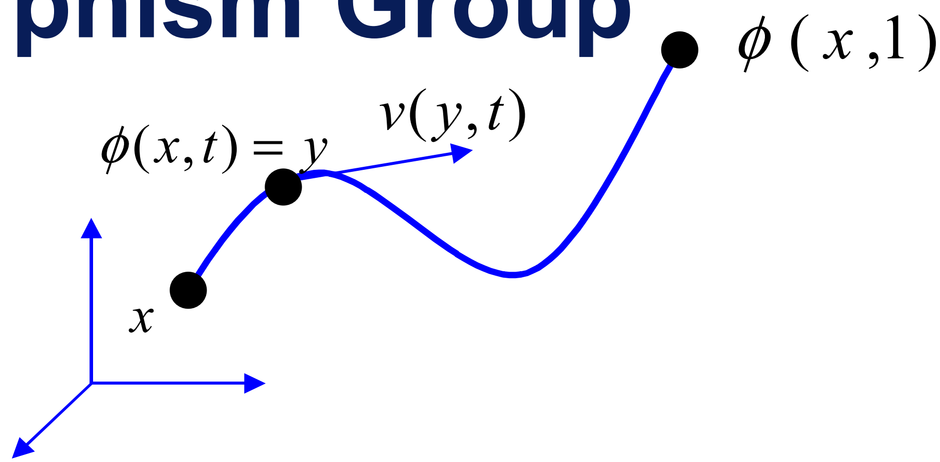
Grenander, Miller, Computational Anatomy: An Emerging Discipline
Quarterly Applied Math. 1998

How do we generate the diffeomorphisms?



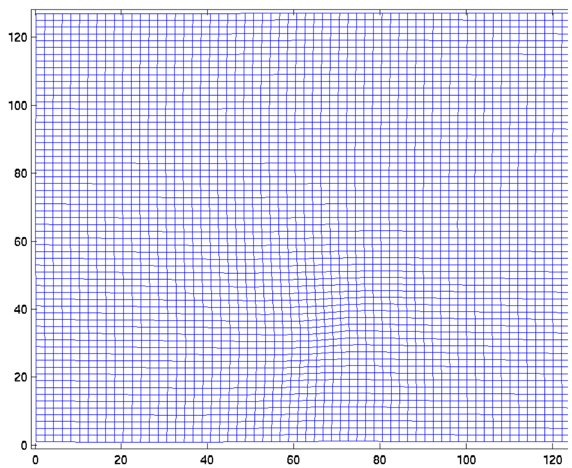
Lagrangian and Eulerian generation of the Diffeomorphism Group





Lagrangian $$\frac{d\phi(x,t)}{dt} = v(\phi(x,t),t) \qquad \phi(x,0)=x$$

Eulerian $$\frac{d\phi^{-1}(x,t)}{dt} = -\nabla\phi^{-1}(x,t)v(x,t) \qquad \phi^{-1}(x,0)=x$$



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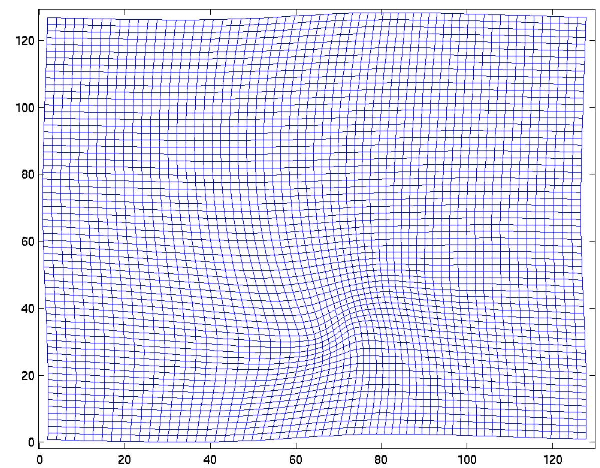

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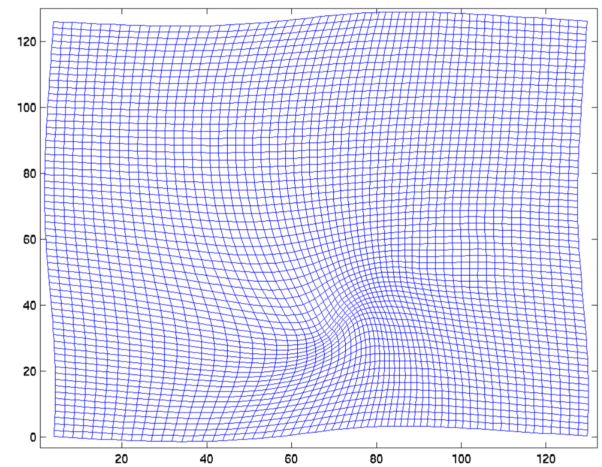

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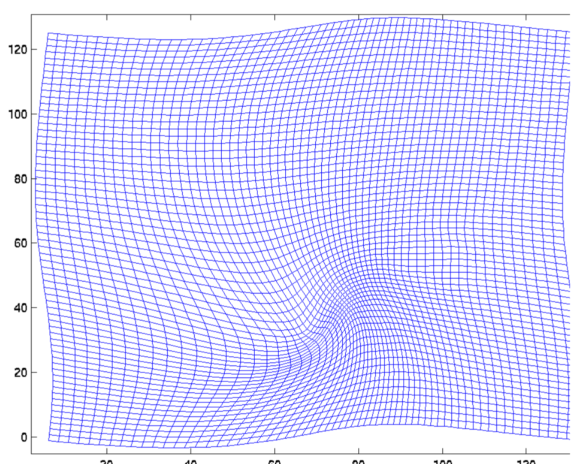
$$\phi_t(x) = \int_0^t v_s(\phi_s(x))ds$$



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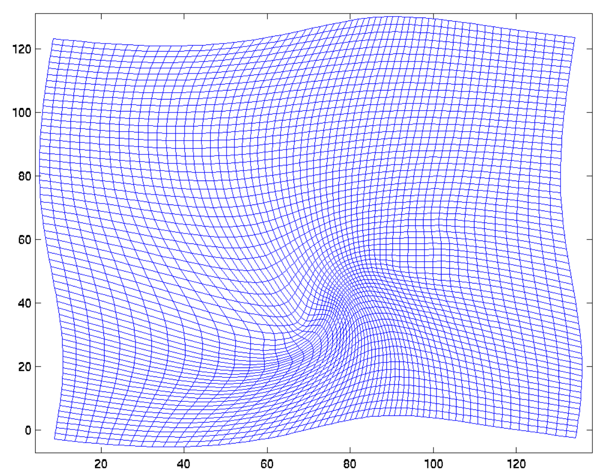

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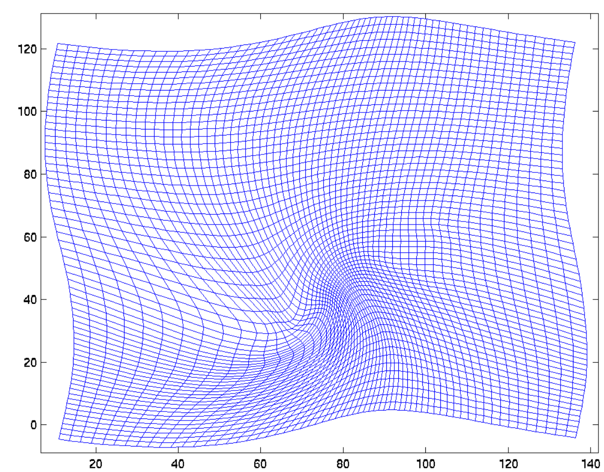

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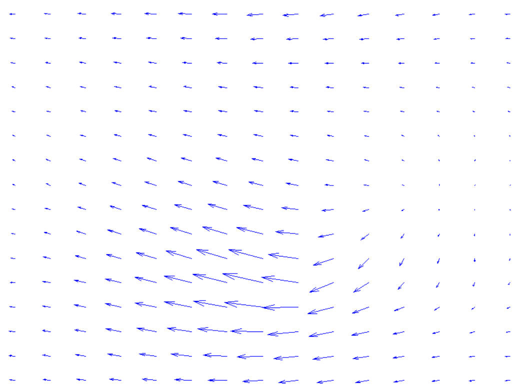

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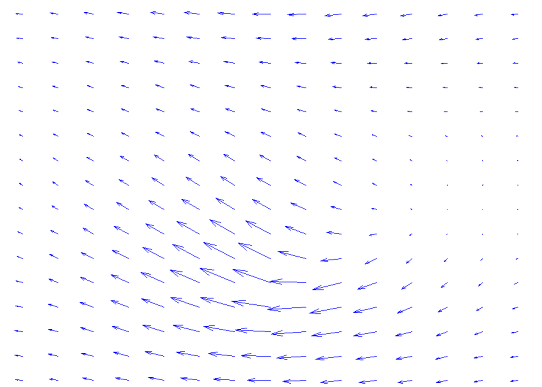

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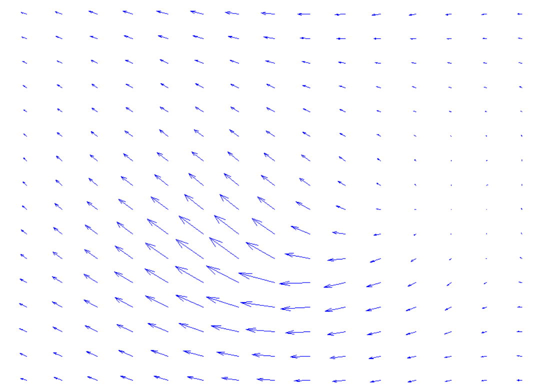

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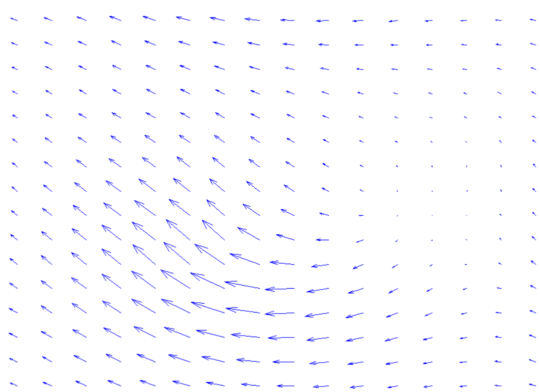
$$\dot{\phi}_t(x) = v_t(\phi_t(x))$$



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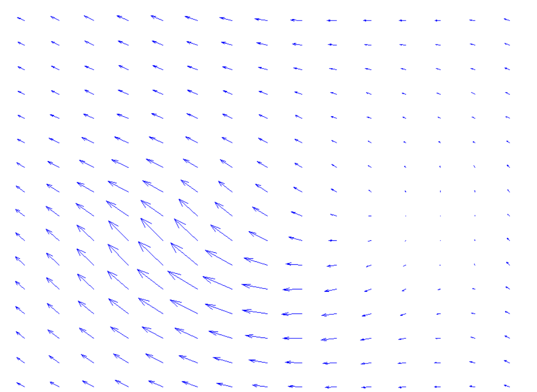

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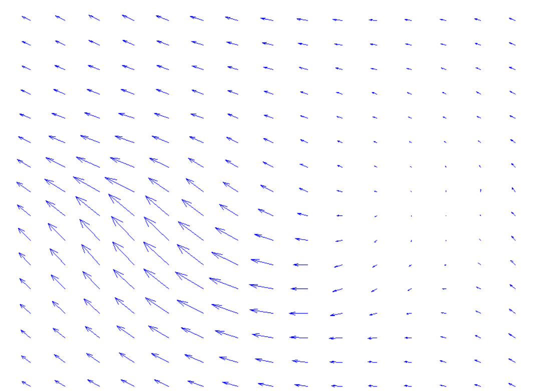

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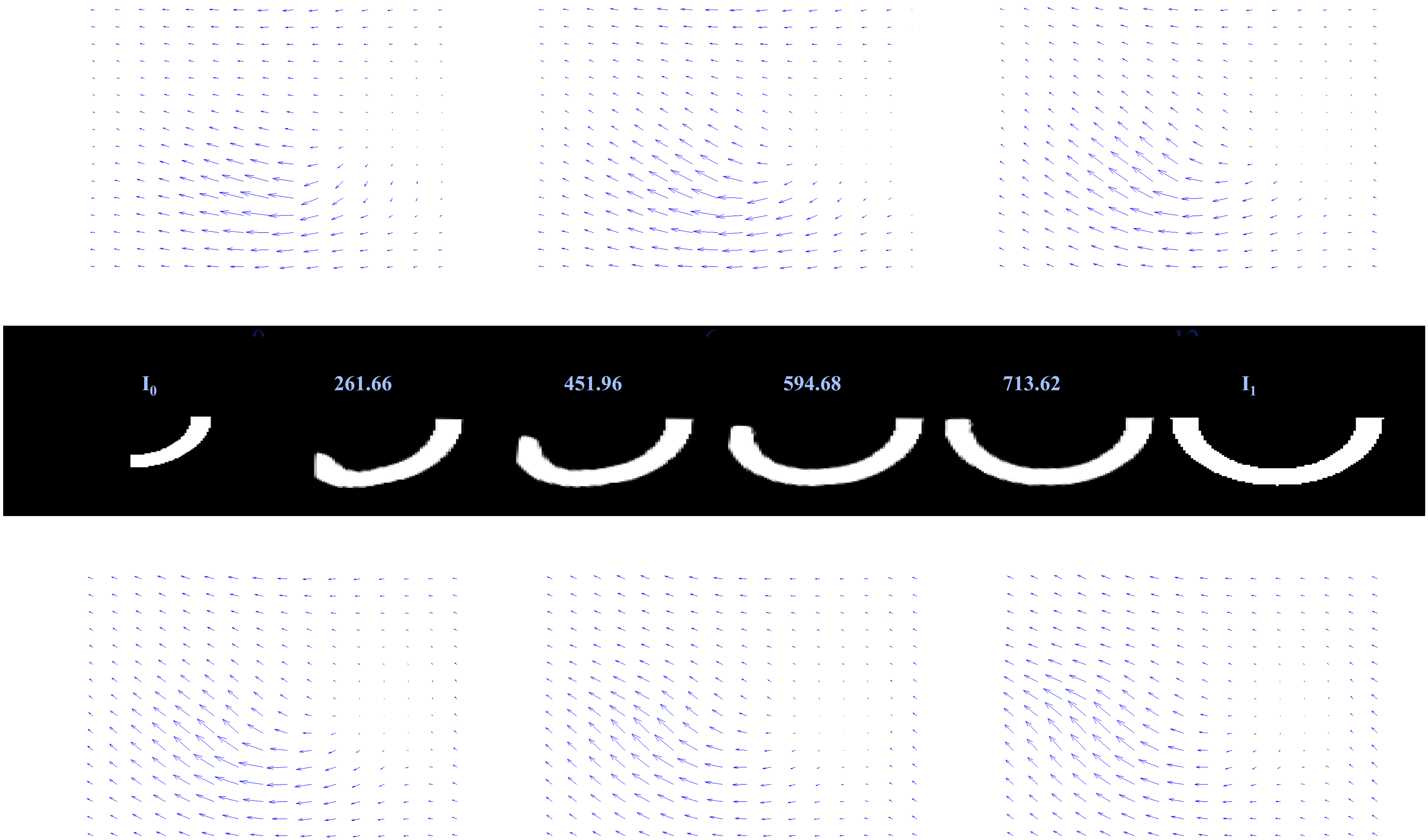
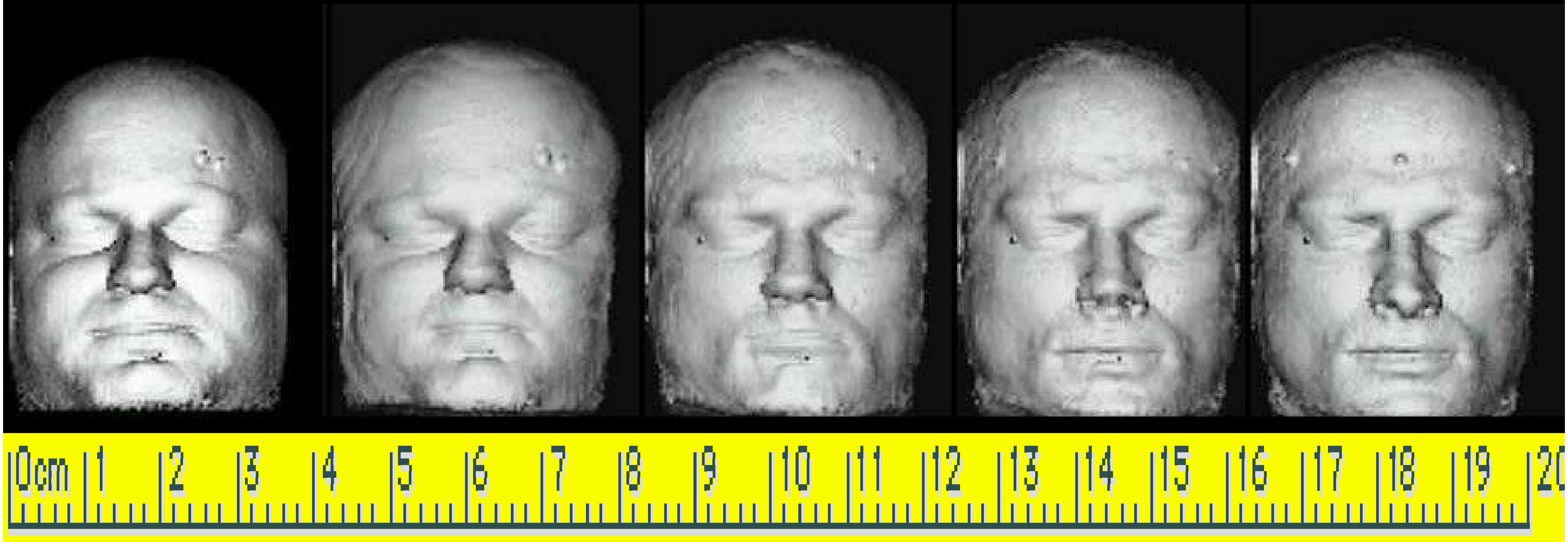
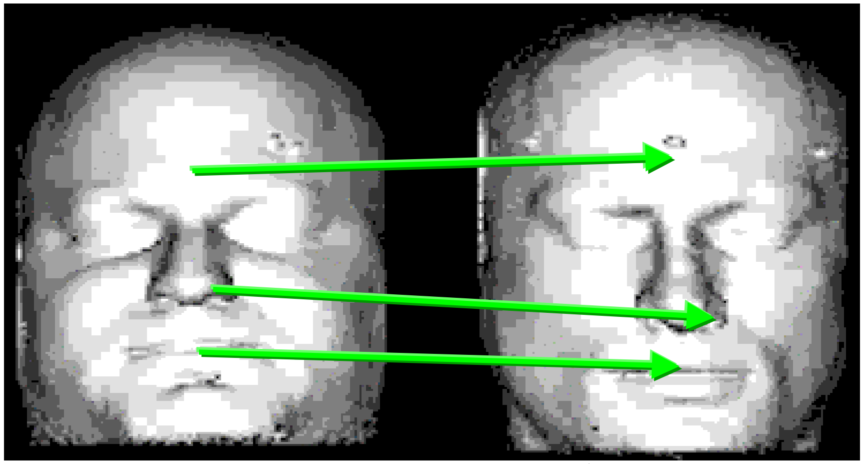

I_0
261.66
451.96
594.68
713.62
I_1

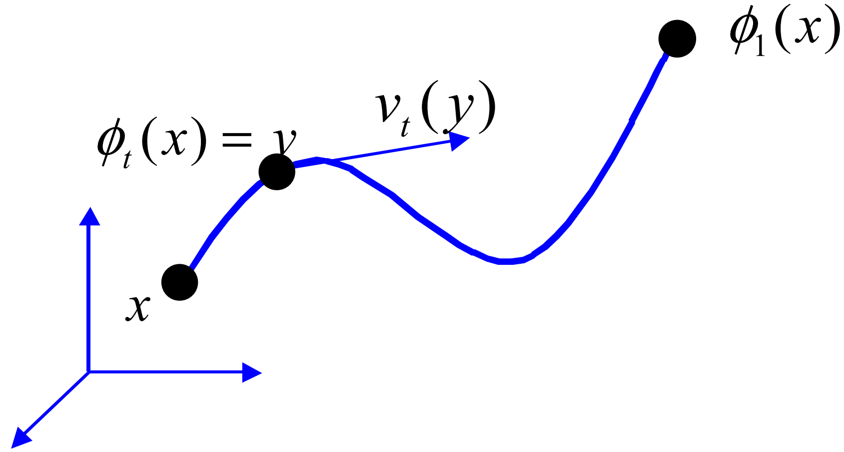
How do we make the orbit into a metric space (Riemannian Manifold)?



Metric distance is the length of the diffeomorphism.

Metric is Geodesic Length





$$\rho(I_0,I_1)=inf_{v\in V}\int_0^1 \| v_t \|_L \, dt$$

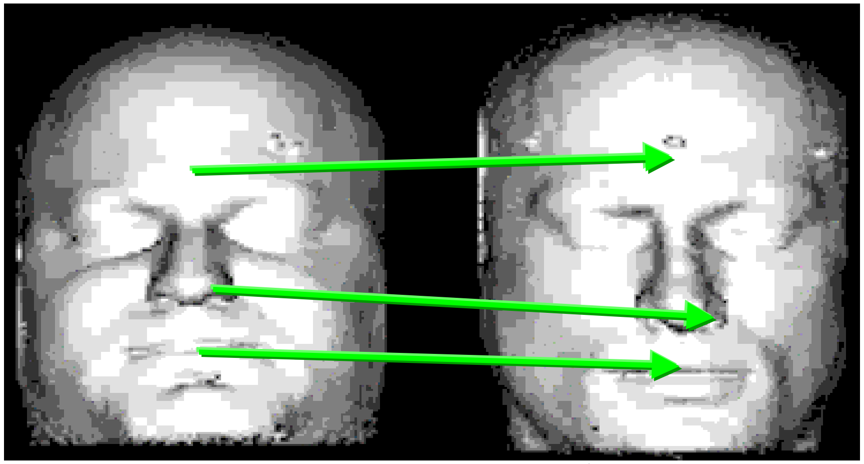
Riemannian Length (metric)

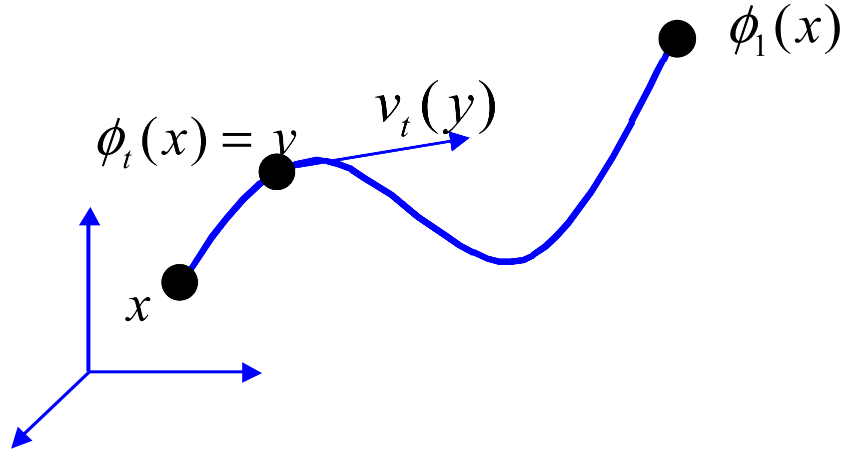
subject to

$$I_0(\phi_1^{-1})=I_1$$

How do we compute the Metric?

Variational Form of the Metric





$$\rho(I_0, I_1) = \inf_{v \in V} \int_0^1 \| v_t \|_L \, dt$$

subject to

$$I_0(\phi_1^{-1}) = I_1$$

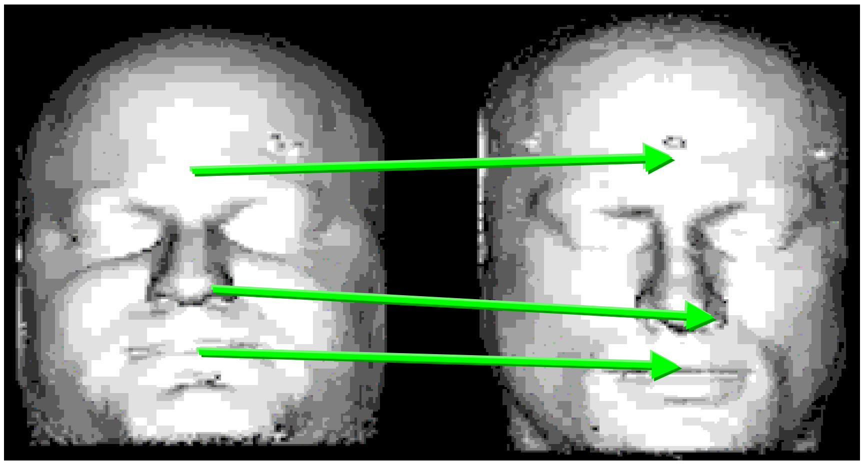
$$\left(\inf_{v \in V} \int_0^1 \| v_t \|_L \, dt \right)^2 = \inf_{v \in V} \int_0^1 \| v_t \|_L^2 \, dt$$

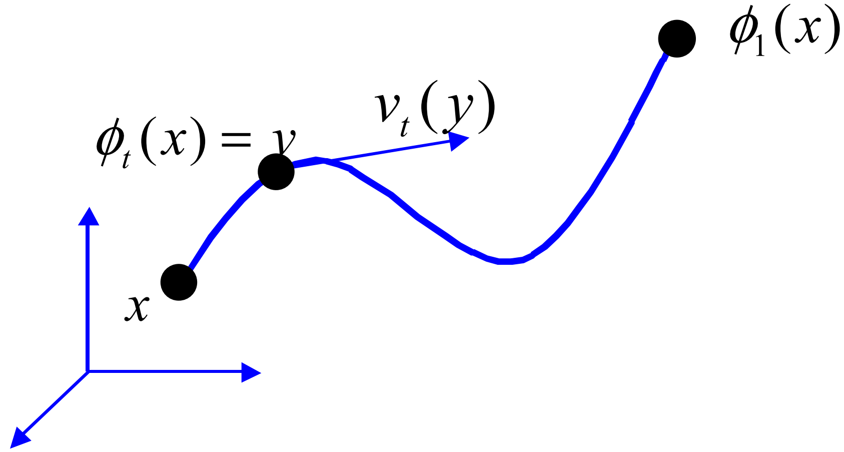
Geodesics are constant speed

Relevant History to Our Work

Landmark matching (small):	**Bookstein 80**
Image matching (small):	**Bajcsy 83, Amit,Gren.,Picc. 90**
Diffeomorphic matching (large):	**Christensen,Rabbitt 93-97**
Computational Anatomy: The Emerging Discipline	***Grenander 1998***
Existence and metrics for estimators on diffeomorphisms:	**Dupuis 97-98** ***Trouve 95-00***
Metrics on photometric and geometric variation:	***Younes,Trouve 2000,2002***
Computing metrics via geodesics	***Beg 2002***
Euler-Lagrange Equations	***Mumford 1998, Trouve-Younes 2002***

Euler-Lagrange Equations





$$\inf_{v \in V} \int_0^1 \| v_t \|_L^2 \, dt$$

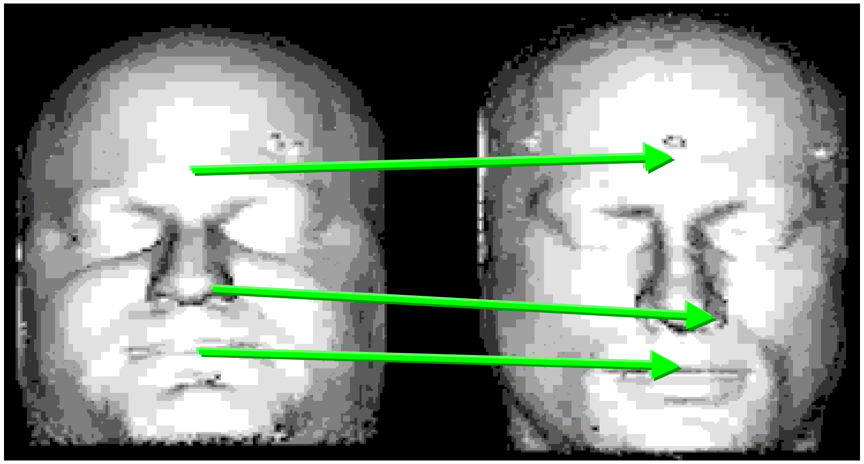
subject to $$I_0(\phi_1^{-1}) = I_1$$

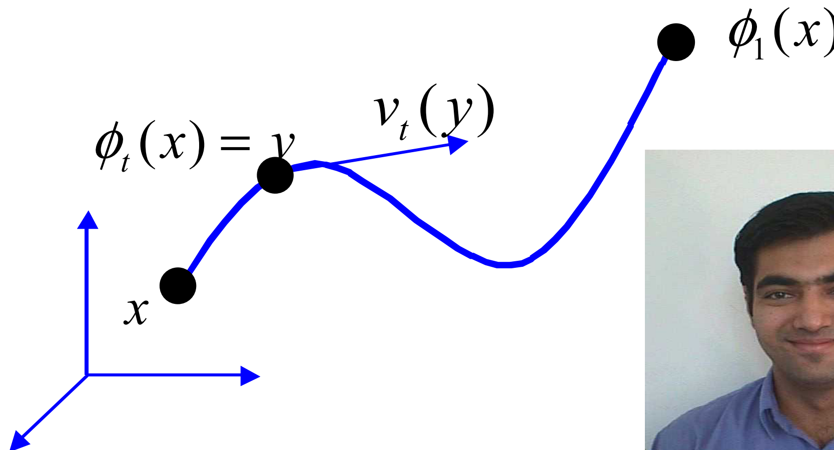
Euler-Lagrange Equations

$$\frac{d}{dt} L v_t + \left(D v_t \right)^* L v_t + \left(D L v_t \right) v_t + \left(\operatorname{div} v_t \right) L v_t = 0$$

Generalizes the finite dimensional Lie group geodesics.
Generalizes Arnold's incompressible flow Euler equation.

Faisal Beg Working in the Vector Fields



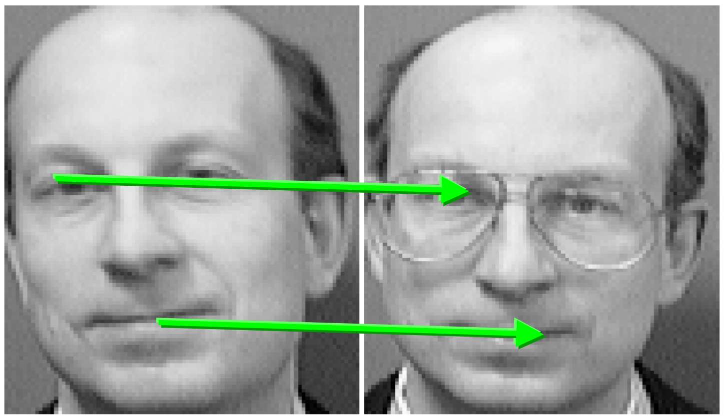


$$\inf_{v\in V}\int_0^1 \| v_t \|_L^2 \, dt + \| I_1 - I_0(\phi_1^{-1}) \|^2$$

Euler-Lagrange Equations

$$\underbrace{Lv_t + (I_1(\phi_1(\phi_t^{-1})) - I_0(\phi_t^{-1}))\nabla(I_0(\phi_t^{-1}))\,|\,D(\phi_1(\phi_t^{-1})\,| = 0}_{Lv_t - |D\phi_t^{-1}|(D\phi_t^{-1})^{*}\, Lv_0(\phi_t^{-1}) = 0}$$

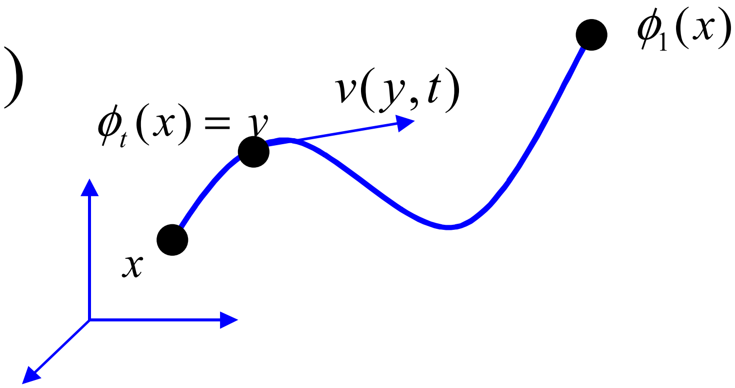
Photometric & Geometric Variation



$$\dot{\phi}_t = v(\phi_t, t)$$

$$I(\phi_t^{-1}, t)$$

$$t \in [0,1]$$



Riemannian Length (metric)

$$\inf_{v,I} \int_0^1 \| v(t) \|_L^2 \, dt + \int_0^1 \| \frac{\partial}{\partial t} I(\phi_t^{-1}, t) \|^2 dt$$

subject to

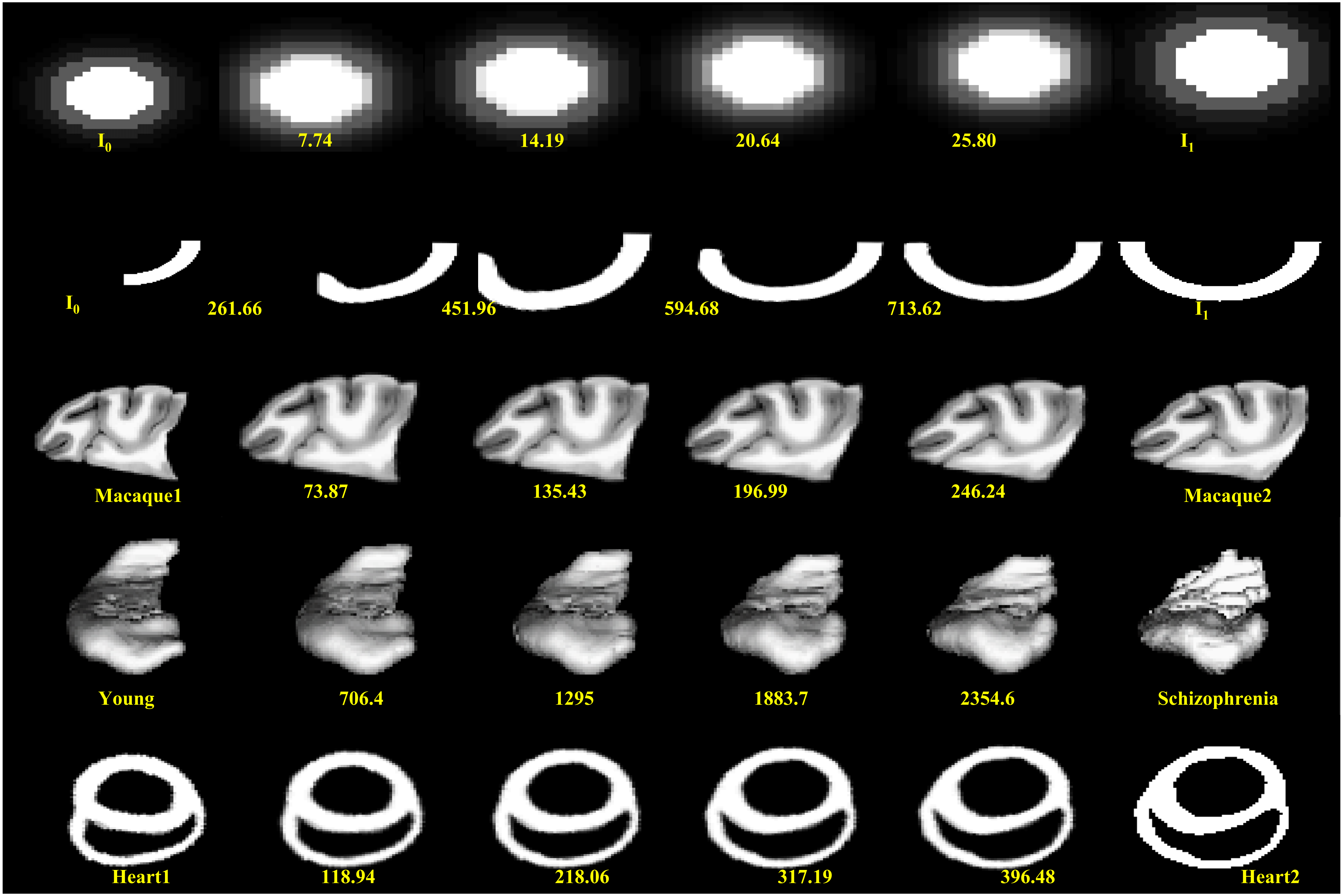
$$I(\phi_0^{-1}, 0) = I_0$$
$$I(\phi_1^{-1}, 1) = I_1$$

$$\frac{d}{dt} Lv(t) + Dv(t)^t Lv(t) + DLv(t)v(t) + div\,v(t) Lv(t) = 0$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} I(\phi_t^{-1}, t)\right) - div\left(\frac{\partial}{\partial t} I(\phi_t^{-1}, t) v(t)\right) = 0$$

Boundary Term t=1

Miller,Younes CVPR 2000



I_0
7.74
14.19
20.64
25.80
I_1
I_0
261.66
451.96
594.68
713.62
I_1
Macaque1
73.87
135.43
196.99
246.24
Macaque2
Young
706.4
1295
1883.7
2354.6
Schizophrenia
Heart1
118.94
218.06
317.19
396.48
Heart2