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# Nonconservative Lagrangian Mechanics: A generalized function approach

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**Abstract**

We reexamine the problem of having nonconservative equations of motion arise from the use of a variational principle. In particular, a formalism is developed that allows the inclusion of fractional derivatives. This is done within the Lagrangian framework by treating the action as a Volterra series. It is then possible to derive two equations of motion, one of these is an advanced equation and the other is retarded.

## 1 Introduction

The problem of having a dissipation term $\dot{q}$ arise in the equations of motion for a system has a long history. Bauer [2] showed that "the equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle". There are loopholes in Bauer's proof, however. One of these is to allow for additional equations of motion to arise. This method was

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employed by Bateman [1]. He used the Lagrangian

$$L = m\dot{x}\dot{y} + \frac{C}{2}\left(x\dot{y} - \dot{x}y\right) \tag{1}$$

which gives the equations of motion

$$m\ddot{x} + C\dot{x} = 0 \qquad m\ddot{y} - C\dot{y} = 0 \tag{2}$$

Bateman's method is not very general, so we look for other methods to model nonconservative systems.

Caldeira and Leggett [6] suggest recognizing that a dissipative system is coupled to an environment. The environment is modelled as a collection of harmonic oscillators which results in the Lagrangian

$$L = \frac{m}{2}\,\dot{q}^2 - V(q) + \sum_{n=1}^{\infty}\left\{\frac{m_n}{2}\,\dot{q}_n^2 - \frac{m_n\omega_n^2}{2}\,(q_n - q)^2\right\} \tag{3}$$

where $q$ is the system's coordinate and the $q_n$'s are the environment's coordinates. While the system by itself is nonconservative, the system plus environment is conservative. This procedure does allow the introduction of very general dissipation terms into the system's equation of motion. However, the microscopic modelling of the environment makes (3) much more complex than, say, (1).

In order to overcome the difficulties of the above two procedures, Riewe examined using fractional derivatives in the Lagrangians [12, 13]. This method takes advantage of another loophole in Bauer's proof. Namely, Bauer assumed that all derivatives were integer ordered. Riewe's method has the advantage of not introducing extra coordinates as in (1) and (3). However, it ultimately results in noncausal equations of motion. A rather ad hoc procedure of replacing anticausal with causal operators needs to be used at the end in order to arrive at causal equations of motion. We will present a method that can be used within Riewe's formalism that avoids this situation.

We propose here a new method of using a variational principle to derive nonconservative equations of motion. Our method is closely related to Riewe's in that we use fractional operators. However, we treat these operators as kernels in a Volterra series. We show that Riewe's formalism can be derived by using certain types of symmetric kernels in the series expansion. A simple modification of the kernels will result in two equations of motion for a system. One of these equations is advanced while the other is retarded, similar to (2).

Our paper is organized as follows. In Section 2 we review fractional integration and differentiation. Riewe's formalism is briefly examined in Section 3. We then give a brief overview of Volterra series in Section 4 before examining our fractional Lagrangian mechanics in Section 5. Section 6 examines the nonconservative harmonic oscillator in a different way than the traditional variational methods. A discussion of some related concepts and future research follows in Section 7.

## 2 Fractional Integration and Differentiation

Fractional integrals and derivatives are generalizations of their usual integer ordered operations. To start developing the theory, let us first write down Cauchy's integral formula

$$f^{(-n)}(t) \quad = \quad \frac{1}{\Gamma(n)} \int_a^t f(\tau)(t-\tau)^{n-1} d\tau \tag{4}$$

where $n > 0$ is an integer, $\Gamma(n)$ is the gamma function, and $a < t$. Equation (4) is a convolution of $f(t)$ and the function

$$\Phi_n^+(t) \quad := \quad \begin{cases} \frac{1}{\Gamma(n)} t^{n-1} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{5}$$

if we set $f(t) \equiv 0$ for $t < a$. So we can rewrite (4) as

$${}_a\mathbf{I}_t^n[f] \quad = \quad f(t) * \Phi_n^+(t) \tag{6}$$

where $*$ is the convolution operation defined by

$$g(t) * h(t) \quad := \quad \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau \tag{7}$$

Equation (6) will be our stepping stone to generalizing the integer ordered operations to fractional order.

The above procedure works so well for the integers $n > 0$, we want to consider extending it to any real $\alpha > 0$. This is obviously possible, so we let

$${}_a\mathbf{I}_t^\alpha[f] \quad = \quad f(t) * \Phi_\alpha^+(t) \tag{8}$$

be the left fractional integral [LFI] of $f(t)$ of order $\alpha > 0$. Everything works fine until we consider the case $\alpha = 0$. We reasonably expect that

$$ {}_a\mathbf{I}_t^0[f] \;=\; f(t) \tag{9} $$

but, it is not immediately obvious that the integral in (6) is not divergent. Also, for $-1 < \alpha < 0$, the integral is obviously divergent. It is apparent that treating $f(t)$ and $\Phi_\alpha^+(t)$ as regular functions will not be sufficiently general for our purposes. Instead we will consider them to be distributions, or generalized functions. [We note that there are other ways to generalize integer ordered derivatives to fractional order [10]. We will work solely with the generalized function approach to interpolate between the integer ordered integrals and derivatives.]

The first order of business is to define the convolution operation for distributions. Let $k(t) = g(t) * h(t)$ and $\varphi(t)$ be a test function. Then [7]

$$ \begin{aligned} \langle k, \varphi \rangle &:= \int k(t)\varphi(t)dt \\ &= \int \left\{ \int g(\xi)h(t-\xi)d\xi \right\} \varphi(t)dt \\ &= \int\int g(\xi)h(\eta)\varphi(\xi+\eta)d\xi d\eta \end{aligned} \tag{10} $$

Equation (10) is meaningful as long as either $g(t)$ or $h(t)$ has bounded support or, $g(t)$ and $h(t)$ are bounded on the same side [e.g., $g(t) \equiv 0$ for $t < t_1$ and $h(t) \equiv 0$ for $t < t_2$]. We will always assume that one of these situations is the case. From (10), it can be seen that the generalization of (7) is

$$ \langle g * h, \varphi \rangle \;=\; \langle g(t), \langle h(\tau), \varphi(t+\tau) \rangle \rangle \tag{11} $$

The convolution operation has the properties

$$ g * h \;=\; h * g \tag{12} $$

$$ f * (g * h) \;=\; (f * g) * h \tag{13} $$

$$ D(g * h) \;=\; (Dg) * h = g * (Dh) \tag{14} $$

where $D(\cdot)$ is the generalized derivative. Remember that the relationship between the generalized and classical derivatives, beginning at $t = a$, is given by [10]

$$ D^n f \;=\; f^{(n)} + \sum_{k=0}^{n-1} \left[ D^{n-k-1}\delta(t-a) \right] f^{(k)}(a) \tag{15} $$

where $f^{(n)}$ is the classical derivative.

Considering $\Phi_\alpha^+(t)$ as a generalized function allows us to extend (8) to any $\alpha$, where the convolution operation is defined as in (11). For $\alpha < 0$, this will define the left fractional derivative [LFD] as

$$\begin{aligned} {}_a\mathbf{D}_t^{-\alpha}[f] &:= {}_a\mathbf{I}_t^{\alpha}[f] \\ &= f(t) * \Phi_\alpha^+(t) \end{aligned} \tag{16}$$

In the sequel, we will find it easier to assume $\alpha > 0$ and use the notation

$$ {}_a\mathbf{D}_t^{\alpha}[f] = f(t) * \Phi_{-\alpha}^+(t) \qquad {}_a\mathbf{D}_t^{-\alpha}[f] = {}_a\mathbf{I}_t^{\alpha}[f] \tag{17}$$

Also, for reasons that will become apparent shortly, we will often set $f(t) \equiv 0$ for $t < a$ and $t > b$, where $a < b$. We do not want any resulting discontinuities in $f(t)$ at $t = b$ to affect the LFDs. So $t$ must be restricted to the interval $a \le t < b$ in the LFDs. It would perhaps be better to write (17) as

$$ {}_{a^-}\mathbf{D}_{t^-}^{\alpha}[f] = \frac{1}{\Gamma(-\alpha)} \int_{a^-}^{t^-} f(\tau)(t-\tau)^{-(\alpha+1)} d\tau \tag{18}$$

To avoid cluttering our notation, we will continue to use the notation in (17) with the understanding that it formally means (18).

The distributions $\Phi_\alpha^+(t)$ have been well studied [7, 10]. Their two most important properties are

$$\Phi_n^+(t) = D^{-n}\delta(t^+) \tag{19}$$

for any integer $n$, and, for any $\beta$ and $\gamma$,

$$\Phi_\beta^+(t-a) * \Phi_\gamma^+(t) = \Phi_{\beta+\gamma}^+(t-a) \tag{20}$$

Equation (20) implies

$$ {}_a\mathbf{D}_t^{\beta}\left[{}_a\mathbf{D}_t^{\gamma}[f]\right] = {}_a\mathbf{D}_t^{\beta+\gamma}[f] \tag{21}$$

$$ {}_a\mathbf{D}_t^{\beta}\left[{}_a\mathbf{D}_t^{-\beta}[f]\right] = f \tag{22}$$

Now let $0 \le n-1 \le \alpha < n$. Then, using (12) – (14) and (19) and (20), we have

$$\begin{aligned} {}_a\mathbf{D}_t^{\alpha}[f] &= f(t) * \Phi_{-\alpha}^+(t) \\ &= f(t) * \left(D^n \Phi_{n-\alpha}^+(t)\right) \\ &= (D^n f(t)) * \Phi_{n-\alpha}^+(t) \qquad (23) \\ &= D^n\left(f(t) * \Phi_{n-\alpha}^+(t)\right) \qquad (24) \end{aligned}$$

Equations (23) and (24) are the distributional forms of the Caputo and Riemann-Liouville fractional derivative, respectively [10]. In the standard definitions of these derivatives, $D^n$ is replaced with $(d/dt)^n$.

In addition to the left fractional operations, we can also define right fractional operations. If we set $f(t) \equiv 0$ for $t > b$ and define

$$\Phi_\alpha^-(t) \quad := \quad \begin{cases} \frac{1}{\Gamma(\alpha)}\,(-t)^{\alpha-1} & t < 0 \\ 0 & t \geq 0 \end{cases} \tag{25}$$

the right fractional operations are defined by

$${}_t\mathbf{D}_b^\alpha\,[f] \quad := \quad f(t) * \Phi_{-\alpha}^-(t) \tag{26}$$

Most of the above observations for the left fractional operations also hold for the right ones. However, (19) needs to be replaced with

$$\Phi_n^-(t) \quad = \quad (-1)^n D^{-n}\delta(t^-) \tag{27}$$

for any integer $n$. When $f(t) \equiv 0$ for $t < a$ and $t > b$, we do not allow any resulting discontinuities in $f(t)$ at $t = a$ to affect the RFDs. Similar to the case for the LFDs, we will take (26) as meaning

$${}_{t^+}\mathbf{D}_{b^+}^\alpha\,[f] \quad = \quad \frac{1}{\Gamma(-\alpha)} \int_{t^+}^{b^+} f(\tau)(\tau - t)^{-(\alpha+1)} \tag{28}$$

though we will continue to use the notation in (26).

Note that for the left operations, the "left" integration limit $a$ determines the allowable functions in the operation ${}_a\mathbf{D}_t^\alpha\,[f]$. Namely, $f(t)$ must vanish for $t < a$. Also, ${}_a\mathbf{D}_t^\alpha\,[f]$ is a function of $\alpha$ and $t$ and, a functional of $f(t)$. Similar comments hold for the right operations. Here, the "right" integration limit $b$ means $f(t) \equiv 0$ for $t > b$. Now let $f(t)$ be compactly supported on the interval $[a, b]$. Then ${}_a\mathbf{D}_t^\alpha\,[f] = 0$ whenever $t < a$. However, ${}_a\mathbf{D}_t^\alpha\,[f]$ does not generally vanish for $t > a$. Thus, the left operations are causal or retarded. Conversely, ${}_t\mathbf{D}_b^\alpha\,[f] = 0$ whenever $t > b$ but, generally, ${}_t\mathbf{D}_b^\alpha\,[f] \neq 0$ for $t < b$. Hence, the right operations are anti-causal or advanced.

Our fractional derivatives satisfy an integration by parts formula. First, assume that $f(t) \equiv 0$ for $t < a$ and $g(t) \equiv 0$ for $t > b$. Then, for any $\beta$,

$$\langle g(\tau)\Phi_\beta^+(\tau - t)f(t), \varphi(t,\tau)\rangle \quad = \quad \langle g(\tau)\Phi_\beta^-(t - \tau)f(t), \varphi(t,\tau)\rangle \tag{29}$$

Hence,

$$\langle g(\Phi_\beta^+ * f), \varphi\rangle \;=\; \langle (g * \Phi_\beta^-) f, \varphi\rangle \tag{30}$$

or

$$\int {}_a\mathbf{D}_t^\beta\,[f]\,g(t)dt \;=\; \int {}_t\mathbf{D}_b^\beta\,[g]\,f(t)dt \tag{31}$$

We note that Riewe's derivation of an integration by parts formula [13, Equation (16)] is flawed on two points. First, the boundary conditions are generally fractional, not integer, ordered. Also, Riewe incorrectly exchanges the classical Caputo derivative [(23) with $D^n$ replaced with $(d/dt)^n$] for the Riemann-Liouville derivative in (24). Fortunately, when vanishing boundary conditions are assumed, these defects are inconsequential. Also notice that (31) implies that any integration by parts inherently introduces time reversal.

When we examine Riewe's fractional mechanics in Section 3, (31) will lead to equations of the form

$$\begin{aligned} \Phi_\beta^- * \left(\Phi_\beta^+ * f\right) &= {}_t\mathbf{D}_b^\alpha\left[{}_a\mathbf{D}_t^\alpha\,[f]\right] \\ &\stackrel{?}{=} \left(\Phi_\beta^- * \Phi_\beta^+\right) * f \end{aligned} \tag{32}$$

The difficulty with (32) is that neither are $\Phi_\beta^+(t)$ or $\Phi_\beta^-(t)$ compactly supported, generally, nor are they bounded on the same side. So we need to make sense of the convolution in (32). To give meaning to the convolution, let us note that the Fourier transform of $\Phi_\beta^+(t)$ is given by [7]

$$\Phi_\beta^+(t) \;\stackrel{\mathcal{F}}{\longleftrightarrow}\; \frac{\exp[\mathrm{sgn}(\omega)i\beta\pi/2]}{|\omega|^\beta} \tag{33}$$

and for $\Phi_\beta^-(t)$

$$\Phi_\beta^-(t) \;\stackrel{\mathcal{F}}{\longleftrightarrow}\; \frac{\exp[-\mathrm{sgn}(\omega)i\beta\pi/2]}{|\omega|^\beta} \tag{34}$$

[Note that (33) and (34) imply that, up to a sign, the fractional derivatives go to the integer ordered derivatives when $\beta$ is an integer.] Then,

$$\Phi_\beta^-(t) * \Phi_\beta^+(t) \;\stackrel{\mathcal{F}}{\longleftrightarrow}\; |\omega|^{-2\beta} \tag{35}$$

Now,

$$\frac{|t|^{2\beta-1}}{2\cos(\beta\pi)\Gamma(2\beta)} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad |\omega|^{-2\beta} \tag{36}$$

We will define

$$\begin{aligned} \Psi_{2\beta}(t) &:= \Phi_{\beta}^{-}(t) * \Phi_{\beta}^{+}(t) \\ &= \frac{|t|^{2\beta-1}}{2\cos(\beta\pi)\Gamma(2\beta)} \end{aligned} \tag{37}$$

and let

$$\Phi_{\beta}^{-}(t) * \Phi_{\beta}^{+}(t) * f(t) \quad \equiv \quad \Psi_{2\beta}(t) * f(t) \tag{38}$$

for any $\beta$ where $f(t) \equiv 0$ for $t < a$ and $t > b$. We call (38) a Feller fractional derivative [FFD] [11] and write this as

$$\begin{aligned} {}_a^t\mathbf{F}_b^{2\alpha}[f] &:= \Psi_{-2\alpha}(t) * f(t) \\ &= {}_t\mathbf{D}_b^{\alpha}\left[{}_a\mathbf{D}_t^{\alpha}\left[f\right]\right] \end{aligned} \tag{39}$$

Note that, for $n$ an integer,

$${}_a^t\mathbf{F}_b^{2n}[f] \quad = \quad (-1)^n f^{(2n)}(t) \tag{40}$$

for $0 < t < T$, but

$${}_a^t\mathbf{F}_b^{2n+1}[f] \quad \neq \quad \pm f^{(2n+1)}(t) \tag{41}$$

Some care is needed when using the FFDs. Formally we have set $f(t) \equiv 0$ for $t > a$ and $t < b$. However, the LFD only acts on the resulting discontinuities that may be present in $f(t)$ at $t = a$, not at $t = b$. Conversely, the RFD acts on the discontinuities at $t = b$, not $t = a$. It is perhaps better to write (37) as

$$\Psi_{2\beta}(t) \quad = \quad \frac{1}{2\cos(\beta\pi)} \left[\Phi_{2\beta}^{+}(t) + \Phi_{2\beta}^{-}(t)\right] \tag{42}$$

Then (39) can be written as

$${}_a^t\mathbf{F}_b^{2\alpha}[f] \quad = \quad \frac{1}{2\cos(\beta\pi)} \left\{{}_a\mathbf{D}_t^{2\alpha}\left[f\right] + {}_t\mathbf{D}_b^{2\alpha}\left[f\right]\right\} \tag{43}$$

We will take (39) as implying (43).

In general, the fractional derivatives are nonlocal in time. That is, they have a "memory". For integer ordered LFDs and RFDs, this memory disappears [i.e., they are "amnesiac"] and they act locally in time. Even integer ordered FFDs are also amnesiac since the kernels $\Psi_{-2n}(t)$ equal, up to a sign, $\Phi^{+}_{-2n}(t)$ and $\Phi^{-}_{-2n}(t)$ in this case. All of the fractional derivatives have a fading memory, however [4]. That is, they are affected more by the recent past and/or future than the distant past and/or future.

# 3 Riewe's Fractional Lagrangian Mechanics

Here we examine Riewe's fractional mechanics [12, 13], restricting our attention to Lagrangian mechanics with Lagrangians of the form

$$L(q, {}_a q_t^{\alpha}, {}_a q_t^{1}) \;=\; \frac{m}{2}\left({}_a q_t^{1}\right)^2 + \frac{C}{2}\left({}_a q_t^{\alpha}\right)^2 - V(q) \tag{44}$$

where $q$ is our [generalized] coordinate, $C$ is a constant, $0 < \alpha < 1$ and,

$${}_a q_t^{\alpha} \;:=\; {}_a\mathbf{D}_t^{\alpha}\left[q\right] \tag{45}$$

We define the action associated with (44) by

$$S[q] \;:=\; \int_a^b L dt \tag{46}$$

Let us consider perturbations $\eta(t)$ of $q(t)$ where $\eta(t)$ vanishes for $t \leq a$ and $t \geq b$ but is otherwise arbitrary. Then,

$$\begin{aligned} \delta S[q] &= \delta \int_a^b L dt \\ &= \int_a^b \left[L(q+\eta, {}_a q_t^{\alpha} + {}_a\eta_t^{\alpha}, {}_a q_t^{1} + {}_a\eta_t^{1}) - L(q, {}_a q_t^{\alpha}, {}_a q_t^{1})\right] dt \end{aligned} \tag{47}$$

Expanding the perturbed Lagrangian in (47)

$$\begin{aligned} L(q+\eta, {}_a q_t^{\alpha} + {}_a\eta_t^{\alpha}, {}_a q_t^{1} + {}_a\eta_t^{1}) &= L(q, {}_a q_t^{\alpha}, {}_a q_t^{1}) + \\ &\quad \frac{\partial L}{\partial q}\,\eta + \frac{\partial L}{\partial_a q_t^{\alpha}}\,{}_a\eta_t^{\alpha} + \frac{\partial L}{\partial_a q_t^{1}}\,{}_a\eta_t^{1} \end{aligned} \tag{48}$$

and using (48) in (47), we have

$$\begin{aligned}\delta S[q] &= \int_a^b \left\{ \frac{\partial L}{\partial q}\,\eta + \frac{\partial L}{\partial_a q_t^\alpha}\,{}_a\eta_t^\alpha + \frac{\partial L}{\partial_a q_t^1}\,{}_a\eta_t^1 \right\} dt \\ &= \int_a^b \eta \left\{ \frac{\partial L}{\partial q} + {}_t\mathbf{D}_b^\alpha \left[ \frac{\partial L}{\partial_a q_t^\alpha} \right] + {}_t\mathbf{D}_b^1 \left[ \frac{\partial L}{\partial_a q_t^1} \right] \right\} dt \end{aligned} \tag{49}$$

where we used (31) in going to the second equality.

Hamilton's principle states that the actual path that a system follows will be that which causes (49) to vanish. Since $\eta$ is infinitesimal but arbitrary, the bracketed term in (49) must vanish for $\delta S[q]$ to vanish. Hence, our Euler-Lagrange equation is

$$ {}_t\mathbf{D}_b^1 \left[ \frac{\partial L}{\partial_a q_t^1} \right] + {}_t\mathbf{D}_b^\alpha \left[ \frac{\partial L}{\partial_a q_t^\alpha} \right] = -\frac{\partial L}{\partial q} \tag{50}$$

For our Lagrangian in (44), we have the following Euler-Lagrange equation of motion

$$\begin{aligned} {}_t\mathbf{D}_b^1 \left[ m_a q_t^1 \right] + {}_t\mathbf{D}_b^\alpha \left[ C_a q_t^\alpha \right] &= m_a\,{}^t\mathbf{F}_b^2[q] + C_a\,{}^t\mathbf{F}_b^{2\alpha}[q] \\ &= \frac{\partial V}{\partial q} \end{aligned} \tag{51}$$

[From (43), we see that (51) is a two-endpoint equation [3].] If, for example, $V(q) = 1/2m\omega^2 q^2$, (51) can be written as

$$\left[ m\Psi_{-2} + C\Psi_{-2\alpha} - m\omega^2\Psi_0 \right] * q = 0 \tag{52}$$

Notice the appearance of the FFD in (51). It arises because of the integration by parts formula (31). In order to have a strictly causal equation of motion, Riewe suggests considering an infinitesimal time interval $[0, 2\epsilon]$ and then replacing all RFDs with LFDs. This seems unsatisfactory because fractional operators have memory due to their nonlocal [in time] nature. By restricting the time interval to an infinitesimal duration, Riewe is effectively erasing this memory. Also, it is questionable if this will provide an accurate approximation. For example, let our time period be $[0, 2\epsilon]$ and

$$\begin{aligned} f(t) &= \delta(t-\epsilon) \\ &= \Phi_0^+(t-\epsilon) \end{aligned} \tag{53}$$

Then,

$$ {}_a\mathbf{D}_t^{2\alpha}\left[f\right] \;=\; \Phi_{-2\alpha}^{+}(t-\epsilon) \tag{54} $$

but,

$$ {}_a^t\mathbf{F}_b^{2\alpha}[f] \;=\; \Psi_{-2\alpha}(t-\epsilon) \tag{55} $$

Now let $\alpha = 1/2$. Obviously (54) and (55) do not agree for $t < \epsilon$. For $t > \epsilon$ we have that $\Psi_{-1}(t-\epsilon) \neq 0$ while $\Phi_{-1}^{+}(t-\epsilon)$ does vanish.

If we blindly follow the above procedure for (51) we have the resulting equation

$$ m\ddot{q} + C\dot{q} = \frac{\partial V}{\partial q} \tag{56} $$

which is missing a minus sign in front of the derivative of the potential $V$. We could of course recognize that ${}_a^t\mathbf{F}_b^2[q] = -\ddot{q}$ for $a < t < b$ and change the sign of $C$ in (44). Then we would have the correct causal equation of motion with friction

$$ m\ddot{q} + C\dot{q} = -\frac{\partial V}{\partial q} \tag{57} $$

However, this requires that we treat integer ordered derivatives differently, which is not entirely satisfactory.

Instead of using the Lagrangian in (44), let us use

$$ L \;=\; -\frac{m}{2}\left({}_a\mathbf{D}_t^1\left[q\right]\right)\left({}_t\mathbf{D}_b^1\left[q\right]\right) - \frac{C}{2}\left({}_a\mathbf{D}_t^\alpha\left[q\right]\right)\left({}_t\mathbf{D}_b^\alpha\left[q\right]\right) - V(q) \tag{58} $$

If we perturb $q$ by $\eta$ in (58), we have, to first order in $\eta$,

$$ \begin{aligned} \delta L \;=\; & -\frac{m}{2}\,{}_a\mathbf{D}_t^1\left[q\right]{}_t\eta_b^1 - \frac{C}{2}{}_a\mathbf{D}_t^\alpha\left[q\right]{}_t\eta_b^\alpha - \frac{1}{2}\frac{\partial V}{\partial q}\,\eta \\ & -\frac{m}{2}\,{}_t\mathbf{D}_b^1\left[q\right]{}_a\eta_t^1 - \frac{C}{2}{}_t\mathbf{D}_b^\alpha\left[q\right]{}_a\eta_t^\alpha - \frac{1}{2}\frac{\partial V}{\partial q}\,\eta \end{aligned} \tag{59} $$

Then, using (31),

$$ \begin{aligned} \int_a^b \delta L\,dt \;=\; & \underbrace{\int_a^b \frac{\eta}{2}\left\{-m\,{}_a\mathbf{D}_t^2\left[q\right] - C\,{}_a\mathbf{D}_t^{2\alpha}\left[q\right] - \frac{\partial V}{\partial q}\right\}dt}_{\text{retarded}} + \\ & \underbrace{\int_a^b \frac{\eta}{2}\left\{-m\,{}_t\mathbf{D}_b^2\left[q\right] - C\,{}_t\mathbf{D}_b^{2\alpha}\left[q\right] - \frac{\partial V}{\partial q}\right\}dt}_{\text{advanced}} \end{aligned} \tag{60} $$

Now,

$$\delta S[q] \quad = \quad \int_a^b \delta L dt \tag{61}$$

To make $\delta S[q]$ vanish, we will require that the bracketed terms in (60) vanish separately. This gives us two equations of motion

$$m_a \mathbf{D}_t^2 [q] + C_a \mathbf{D}_t^{2\alpha} [q] = -\tfrac{\partial V}{\partial q} \quad \text{(retarded)} \tag{62}$$

$$m_t \mathbf{D}_b^2 [q] + C_t \mathbf{D}_b^{2\alpha} [q] = -\tfrac{\partial V}{\partial q} \quad \text{(advanced)} \tag{63}$$

For the special case $\alpha = 1/2$, (62) and (63) become

$$m\ddot{q} + C\dot{q} = -\tfrac{\partial V}{\partial q} \quad \text{(retarded)} \tag{64}$$

$$m\ddot{q} - C\dot{q} = -\tfrac{\partial V}{\partial q} \quad \text{(advanced)} \tag{65}$$

respectively, for $a < t < b$.

Comparing (64) and (65) with (2), we see that Bateman's method is included in Riewe's formalism provided we use Lagrangians as in (58) and, require the advanced and retarded parts of the perturbed action to vanish separately. [These types of Lagrangians were not considered explicitly by Riewe in [12, 13].] Allowing both a retarded and an advanced equation of motion to arise from the variation of the action seems more natural than, for example, (52). It avoids the final procedure of replacing ${}_t\mathbf{D}_b^\alpha [q]$ with ${}_a\mathbf{D}_t^\alpha [q]$. Also, the Lagrangian in (58) is preferable to that in (44) because it does not apriori assume that the LFDs are to be favored over the RFDs. Now we turn our attention to an alternate way of constructing nonconservative Lagrangians.

# 4 Volterra Series

In order to develop our new formalism of nonconservative Lagrangians, we will need some background on Volterra series [5, 15]. The Volterra series is a generalization to functionals of the power series of a function. For some functional $\mathcal{V}[q]$, we define the symmetric kernels

$$K_n^{(s)}(\tau_1, \ldots, \tau_n) \quad := \quad \frac{\delta^n \mathcal{V}[q]}{\delta q(\tau_1) \cdots \delta q(\tau_n)} \tag{66}$$

The $K_n^{(s)}(\cdot)$'s are symmetric under an interchange of the $\tau_i$'s. So, for example, $K_2^{(s)}(\tau_1, \tau_2) = K_2^{(s)}(\tau_2, \tau_1)$. Introducing the notation

$$K_n^{(s)} \star q^n \quad := \quad \int_{\tau_1} \cdots \int_{\tau_n} K_n^{(s)}(\tau_1, \ldots, \tau_n) q(\tau_n) \cdots q(\tau_1) d\tau_n \cdots d\tau_1 \tag{67}$$

we can expand the functional $\mathcal{V}[q]$ in the Volterra series

$$\mathcal{V}[q] \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n!} K_n^{(s)} \star q^n \tag{68}$$

[For our purposes we can assume that $K_0^{(s)} = \mathcal{V}[0] \equiv 0$.] It is easy to show that

$$\begin{aligned} \frac{\delta K_n^{(s)} \star q^n}{\delta q(t)} &= n K_n^{(s)} \star q^{n-1} \\ &:= n \int_{\tau_2} \cdots \int_{\tau_n} K_n^{(s)}(t, \tau_2, \ldots, \tau_n) \\ &\qquad q(\tau_n) \cdots q(\tau_2) d\tau_n \cdots d\tau_2 \end{aligned} \tag{69}$$

The symmetric kernels are the natural choice to use in a Volterra series. However, we may be given asymmetric kernels and would like to symmetrize them or vice versa. As motivation, consider the function

$$v(\mathbf{q}) \quad = \quad \frac{1}{2} \sum_i K_{ii} q_i^2 + \sum_{i<j} K_{ij} q_i q_j \tag{70}$$

where $\mathbf{q} = [q_1, \ldots, q_n]$. We can symmetrize $v(\mathbf{q})$ into the form

$$v(\mathbf{q}) \quad = \quad \frac{1}{2} \sum_{i,j} K_{ij} q_i q_j \tag{71}$$

where $K_{ij} = K_{ji}$ is a symmetric matrix. We will be particularly interested in triangular kernels given by

$$K_n^{(t)}(\tau_1, \ldots, \tau_n) \quad = \quad 0 \qquad \text{unless } \tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \tag{72}$$

Now, let $\sigma$ be a permutation of $1, \ldots, n$. The symmetrization of (72) is defined as

$$\begin{aligned} \mathrm{sym} K_n^{(t)}(\tau_1, \ldots, \tau_n) &:= \frac{1}{n!} \sum_{\sigma} K_n^{(t)}(\tau_{\sigma_1}, \ldots, \tau_{\sigma_n}) \\ &= \frac{1}{n!} K_n^{(s)}(\tau_1, \ldots, \tau_n) \end{aligned} \tag{73}$$

# 5 Volterra Series Fractional Lagrangian Mechanics

Let us now reconsider the nonconservative harmonic oscillator equation of motion in (52). Using the notation in (69), (52) becomes

$$K_2^{(s)} \star q^1 \quad = \quad 0 \tag{74}$$

where

$$K_2^{(s)}(t,\tau) \quad := \quad m\Psi_{-2}(t-\tau) + C\Psi_{-2\alpha}(t-\tau) - m\omega^2\Psi_0(t-\tau) \tag{75}$$

Let our action be given by

$$\mathcal{V}_2[q] \quad = \quad \frac{1}{2}\, K_2^{(s)} \star q^2 \tag{76}$$

Then,

$$\frac{\delta \mathcal{V}_2[q]}{\delta q(t)} \quad = \quad K_2^{(s)} \star q^1 \tag{77}$$

Requiring (77) to vanish gives us (74).

Suppose now that we have a driven harmonic oscillator

$$m\ddot{q} + m\omega^2 q \quad = \quad f(t) \tag{78}$$

We can form a new functional

$$\mathcal{V}_{2'}[q] \quad = \quad K_{1'}^{(s)} \star q^1 + \frac{1}{2}\, K_{2'}^{(s)} \star q^2 \tag{79}$$

where

$$K_{2'}^{(s)}(t,\tau) \quad := \quad m\Psi_{-2}(t-\tau) - m\omega^2\Psi_0(t-\tau) \tag{80}$$

It immediately follows that, ignoring boundary conditions,

$$\frac{\delta \mathcal{V}_{2'}[q]}{\delta q(t)} \quad = \quad K_{1'}^{(s)}(t) - m\ddot{q} - m\omega^2 q \tag{81}$$

Requiring (81) to vanish and comparing with (78), we see that $K_{1'}^{(s)}(t) = f(t)$. We can also handle higher order potentials. Let, for example,

$$\mathcal{V}_{3'}[q] \quad = \quad K_{1'}^{(s)} \star q^1 + \frac{1}{2!}\, K_{2'}^{(s)} \star q^2 + \frac{1}{3!}\, K_{3'}^{(s)} \star q^3 \tag{82}$$

where, for some constant $C$,

$$K_{3'}^{(s)}(\tau_1,\tau_2,\tau_3) \;:=\; C\Psi_0(\tau_1-\tau_2)\Psi_0(\tau_2-\tau_3) \tag{83}$$

Then, again ignoring boundary terms,

$$\frac{\delta\mathcal{V}_{3'}[q]}{\delta q(t)} \;=\; f(t)-m\ddot{q}-m\omega^2 q+\frac{C}{2}\,q^2 \tag{84}$$

We recognize (82) as the beginning of the Volterra series for some functional $\mathcal{V}[q]$. To all orders of $q$,

$$\mathcal{V}[q] \;=\; \sum_{n=1}^{\infty}\frac{1}{n!}\,K_n^{(s)}\star q^n \tag{85}$$

[We can ignore the $n=0$ term in (85) since this only adds an irrelevant constant to $\mathcal{V}[q]$.] For $n\geq 2$, the $K_n^{(s)}(\cdot)$'s are interpreted as the environment's reaction to $q$, which affects $q$'s evolution. Any forcing function is included in $K_1^{(s)}(t)$.

All of the actions considered above share two key properties:

1. The kernels $K_n^{(s)}(\cdot)$ are all localized along the line $\tau_1=\tau_2$.
2. The kernels satisfy the relation $K_n^{(s)}(\cdot)=K_n^{(t)}(\cdot)$.

These properties make the above actions particularly easy to analyze. However, it is impossible to introduce even the simple term $C\dot{q}$ into the equations of motion using the $\Psi_\alpha$'s [see (41)]. Using triangular, instead of symmetric, kernels results in a more flexible formalism. This amounts to using the $\Phi_\alpha^\pm$'s in the Volterra series instead of the $\Psi_\alpha$'s. We will then be able to construct symmetric kernels that only use the $\Phi_\alpha^\pm$'s, not the $\Psi_\alpha$'s. This requires us to be careful about the boundary terms in our equations. It is this situation that we now turn our attention to.

We return again to the nonconservative harmonic oscillator. For some constant $C$, define the triangular kernels

$$K_2^+(t,\tau) \;:=\; -\Big[m\Phi_{-2}^+(t-\tau)+C\Phi_{-2\alpha}^+(t-\tau)+m\omega^2\Phi_0^+(t-\tau)\Big] \tag{86}$$

$$K_2^-(\tau,t) \;:=\; -\Big[m\Phi_{-2}^-(\tau-t)+C\Phi_{-2\alpha}^-(\tau-t)+m\omega^2\Phi_0^-(\tau-t)\Big] \tag{87}$$

where

$$K_2^+(t,\tau) \;=\; K_2^-(\tau,t) \tag{88}$$

Now consider the functional

$$\widehat{\mathcal{V}}[q] \;:=\; \frac{1}{2}\int_{a^-}^{b^+}\int_{a^-}^{\tau_1^-} K_2^+(\tau_1,\tau_2)q(\tau_2)q(\tau_1)d\tau_2 d\tau_1 \tag{89}$$

The functional derivative of (89) is given by [15]

$$\begin{aligned}
\frac{\delta\widehat{\mathcal{V}}[q]}{\delta q(t)} &= \lim_{h\to 0}\frac{1}{2h}\left\{\int_{a^-}^{b^+}\int_{a^-}^{\tau_1^-} K_2^+(\tau_1,\tau_2)[q(\tau_2)+h\delta(\tau_2-t)]\times\right.\\
&\qquad\qquad [q(\tau_1)+h\delta(\tau_1-t)]d\tau_2 d\tau_1 -\\
&\qquad\left.\int_{a^-}^{b^+}\int_{a^-}^{\tau_1^-} K_2^+(\tau_1,\tau_2)q(\tau_2)q(\tau_1)d\tau_2 d\tau_1\right\}\\
&= \frac{1}{2}\int_{a^-}^{b^+}\int_{a^-}^{\tau_1^-} K_2^+(\tau_1,\tau_2)\delta(\tau_2-t)q(\tau_1)d\tau_2 d\tau_1 +\\
&\quad \frac{1}{2}\int_{a^-}^{b^+}\int_{a^-}^{\tau_1^-} K_2^+(\tau_1,\tau_2)q(\tau_2)\delta(\tau_1-t)d\tau_2 d\tau_1\\
&= \underbrace{\frac{1}{2}\int_{t^+}^{b^+} K_2^-(t,\tau_1)q(\tau_1)d\tau_1}_{\text{advanced}} + \underbrace{\frac{1}{2}\int_{a^-}^{t^-} K_2^+(t,\tau_2)q(\tau_2)d\tau_2}_{\text{retarded}}
\end{aligned} \tag{90}$$

where $a \le t \le b$. Instead of requiring the sum in (90) to vanish, we will require the advanced and retarded parts of the action's variation to vanish separately. This gives us two equations of motion for our system

$$\left[m\Phi_{-2}^+(t) + C\Phi_{-2\alpha}^+(t) + m\omega^2\Phi_0^+(t)\right] * q(t) = 0 \quad \text{(retarded)} \tag{91}$$

$$\left[m\Phi_{-2}^-(t) + C\Phi_{-2\alpha}^-(t) + m\omega^2\Phi_0^-(t)\right] * q(t) = 0 \quad \text{(advanced)} \tag{92}$$

From (90), we see that $q(\tau_1) \equiv 0$ for $\tau_1 > b$ and $q(\tau_2) \equiv 0$ for $\tau_2 < a$ in (89).

Note that if our kernels only contain terms $\Phi_{2n}^{\pm}$, $n$ an integer, requiring the advanced and retarded parts to vanish separately is equivalent to requiring the sum in (90) to vanish, ignoring boundary conditions. This is because $\Phi_{2n}^+ = \Phi_{2n}^-$ and both equal, up to a sign, $\Psi_{2n}$. So in this case we can freely use the symmetric kernels $\Psi_{2n}$ in our action. We can also extend the above action to a driven harmonic oscillator and higher order potentials, as we did earlier. Again, this is due to the fact that $\Psi_0 = \Phi_0^{\pm}$ and also that $K_1^{(s)} = K_1^{\pm}$.

The kernel in (89) is lower triangular in the $\tau_1\tau_2$-plane [i.e., $K_2^+(\tau_1,\tau_2) \equiv 0$ when $\tau_1 \le \tau_2$]. We could have equally well used the functional

$$\widetilde{\mathcal{V}}[q] \;:=\; \frac{1}{2}\int_{a^-}^{b^+}\int_{\tau_1^+}^{b^+} K_2^+(\tau_2,\tau_1)q(\tau_2)q(\tau_1)d\tau_2 d\tau_1 \tag{93}$$

to arrive at the equations of motion in (91) and (92). Here the kernel is upper triangular in the $\tau_1\tau_2$-plane [i.e., $K_2^+(\tau_2,\tau_1) \equiv 0$ when $\tau_1 \geq \tau_2$]. A derivation similar to that in (90) shows that, if we use (93) for our action , then $q(\tau_1) \equiv 0$ for $\tau_1 < a$ and $q(\tau_2) \equiv 0$ for $\tau_2 > b$. It follows that the symmetric action

$$\begin{aligned} \mathcal{V}[q] &:= \frac{1}{2}\left\{\widehat{\mathcal{V}}[q] + \widetilde{\mathcal{V}}[q]\right\} \\ &= \frac{1}{2}\int_{a^-}^{b^+}\int_{a^-}^{b^+}\left\{\frac{1}{2}\left[K_2^+(\tau_1,\tau_2) + K_2^+(\tau_2,\tau_1)\right]\right\} q(\tau_2)q(\tau_1)d\tau_2 d\tau_1 \end{aligned} \tag{94}$$

could also be used to derive (91) and (92), where $q(\tau_i) \equiv 0$, $i = 1, 2$, for $\tau_i < a$ and $\tau_i > b$. The above is easier to see if we let $K_2^+(t) := K_2^+(t, 0)$ and $K_2^-(t) := K_2^-(t, 0)$. Then (89) is given by

$$\widehat{\mathcal{V}}[q] = \frac{1}{2}\int q(t)\left[K_2^+(t) * q(t)\right] dt \tag{95}$$

Using the integration by parts formula in (30) gives us (93)

$$\widetilde{\mathcal{V}}[q] = \frac{1}{2}\int q(t)\left[K_2^-(t) * q(t)\right] dt \tag{96}$$

Adding (95) to (96), and multiplying by $1/2$, results in (94)

$$\mathcal{V}[q] = \frac{1}{2}\int q(t)\left\{\frac{1}{2}\left[K_2^+(t) + K_2^-(t)\right] * q(t)\right\} dt \tag{97}$$

Let us now collect some remaining observations. The usual action for the harmonic oscillator is given by

$$\begin{aligned} S[q] &= \frac{1}{2}\int\left[m\dot{q}^2 - m\omega^2 q^2\right] dt \\ &= -\frac{1}{2}\int q\left[m\ddot{q} + m\omega^2 q\right] dt + \frac{1}{2}\, q\dot{q}\Big|_a^b \end{aligned} \tag{98}$$

where we used an integration by parts in the second equality. The Volterra series action in (95), with $C = 0$ in (86), gives

$$\begin{aligned} \widehat{\mathcal{V}}[q] &= -\frac{1}{2}\int q\left[m\Phi_{-2}^+ + m\omega^2\Phi_0^+\right] * q\, dt \\ &= -\frac{1}{2}\int q\left[m\ddot{q} + m\omega^2 q\right] dt - \frac{1}{2}\int q\left[\dot{q}(a)\delta(t-a) + q(a)\dot{\delta}(t-a)\right] dt \\ &= -\frac{1}{2}\int q\left[m\ddot{q} + m\omega^2 q\right] dt \end{aligned} \tag{99}$$

where we used (15) for the second equality. Hence,

$$S[q] \;=\; \widehat{\mathcal{V}}[q] + \frac{1}{2}\, q\dot{q}\Big|_a^b \tag{100}$$

Thus, $\widehat{\mathcal{V}}[q]$ differs from $S[q]$ only by the boundary terms of $q(t)$, which, by our above analysis, does not affect the resulting equation of motion. This also holds for (96) and (97). Returning to (95), let us perturb $q(t)$ by $\eta(t)$. Then,

$$\delta\widehat{\mathcal{V}}[q] \;=\; \underbrace{\frac{1}{2}\int q\left[K_2^+ * \eta\right] dt}_{\text{advanced}} + \underbrace{\frac{1}{2}\int \eta\left[K_2^+ * q\right] dt}_{\text{retarded}} \tag{101}$$

The second term on the right of (101) is what we typically want in order to derive our equation of motion for $q(t)$. However, the first term on the right of (101) is interesting. It shows that the advanced equation of motion for $q(t)$ arises because of the perturbation of the environment's reaction due to $\eta(t)$. That is, using (30),

$$\int q\left[K_2^+ * \eta\right] dt \;=\; \int \eta\left[K_2^- * q\right] dt \tag{102}$$

So it seems that the future evolution of $q(t)$ is affected by its past evolution because of the memory "stored" in the environment.

# 6 Perturbing the Environment

So far we have examined everything in a fairly standard way. We assumed that the environment is described by $K_2^+(t) := K_2^+(t,0)$ [see (86)] and introduced a particle into this environment via $q(t)$. Then we perturbed the particle's path and required that the variation in the action vanish under this perturbation. Notice that the particle did not change the environment's kernel given by $K_2^+(t)$. So the particle itself must be so negligible that the environment's kernel does not substantially change under its introduction. That is, the particle is a perturbation to the environment. Let us explore this idea more for the nonconservative harmonic oscillator. This will lead to a more holistic view of mechanics which ignores the distinction between the environment and the system, in the case of the harmonic oscillator.

Let us assume that the environment is adequately described by the generalized function [see (20) for the second equality below]

$$\begin{aligned} K_2^+(t) &:= m\Phi_{-2}^+(t) + C\Phi_{-2\alpha}^+(t) + m\omega^2\Phi_0^+(t) \\ &= \left[m\Phi_{-2}^+(t) + C\Phi_{-2\alpha}^+(t) + m\omega^2\Phi_0^+(t)\right] * \widehat{\Phi}_0^+(t) \\ &= K_2^+(t) * \widehat{\Phi}_0^+(t) \end{aligned} \tag{103}$$

where the hat on $\widehat{\Phi}_0^+(t)$ is for bookkeeping purposes only. Now let us perturb the environment $K_2^+(t)$ by perturbing $\widehat{\Phi}_0^+(t)$ by $\widehat{\eta}(t)$, where $\widehat{\eta}(t)$ is infinitesimal in the, e.g., $L^2$-norm compared to $\widehat{\Phi}_0^+(t) = \widehat{\delta}(t)$. In particular, we will not require that $\widehat{\eta}(t)$ vanish at any boundaries. Then,

$$\delta K_2^+(t) = K_2^+(t) * \widehat{\eta}(t) \tag{104}$$

Requiring (104) to vanish gives us exactly (91) when we identify $\widehat{\eta}(t) \equiv q(t)$.

In (103), we assumed that the environment reacts causally to any perturbation. This resulted in the retarded equation of motion in (104). If, instead, we considered the kernel $K_2^-(t) := K_2^-(0,t)$, [see (87)], then the advanced equation of motion in (92) would have resulted instead of (104). So, we see that it is not necessary, for the nonconservative harmonic oscillator, to construct an action in order to derive the equations of motion. How far this idea can be advanced to more general systems is an open question.

# 7 Discussion

Let us look at the retarded equation in (91) a little. This is a convolution between the coordinate $q(t)$ and the distribution $K_2^+(t,0) := K_2^+(t)$. An insightful way of viewing this is to think of $K_2^+(t)$ as the environment's response "function", where, for an arbitrary $f(t)$,

$$y(t) = K_2^+(t) * f(t) \tag{105}$$

Then, the actual paths that a system can follow will be those $f(t)$ such that $y(t) \equiv 0$ in (105). That is, the path a system follows will be those such that the environment's response to it vanishes. This treatment of fractional derivatives as signal processors is well known [14] and can be extended to more general Volterra series than that

in (91) [4, 9]. Similar comments hold for (92) and its generalizations, but, in this case, the environment's response is anti-causal. We have not pursued this line of research here. However, it does open up the possibility of examining mechanics from a systems theoretic viewpoint [see, e.g., [8] for an introduction to systems theory].

We have not considered the most general action here. Instead, our attention was restricted to including fractional derivatives in an equation of motion derived by using a variational principle. In this respect we have succeeded. Further research is needed to see how far our formalism can be developed and, how useful it will be in situations other than those considered here. In particular, it would be interesting to extend the formalism in Section 6 to more general situations.

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